REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From – To)*
11-10-2007	Final Report	1 September 2004 - 1 September 07

4. TITLE AND SUBTITLE

The Distribution Function and Transport Properties of Charged Particles in Strong, Crossed Electric and Magnetic fields

5a. CONTRACT NUMBER
FA8655-03-D-0001, Delivery Order 0017

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)

Professor Andrei Ender

5d. PROJECT NUMBER

5d. TASK NUMBER

5e. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Ioffe Institute
Politeknicheskaya St. 26
St. Petersburg 194021
Russia

8. PERFORMING ORGANIZATION REPORT NUMBER

N/A

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

EOARD
Unit 4515 BOX 14
APO AE 09421

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)
EOARD Task 04-9002

12. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT

This report results from a contract tasking Ioffe Institute as follows: The contractor will investigate the Distribution Function and Transport Properties of Charged Particles in Strong, Crossed Electric and Magnetic fields. The grantee will investigate a method to calculate the matrix elements of the Boltzmann equation collision integral for large values of the indices as well as arbitrary interaction potentials. This proposal is targeted for solving the transport problems in strong fields and would be very useful in the computation of weakly ionized plasmas and the extraction of cross- section data from experiments.

15. SUBJECT TERMS
EOARD, Magneto Hydro Dynamic (MHD)

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18, NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			SURYA SURAMPUDI
UNCLAS	UNCLAS	UNCLAS	UL	59	19b. TELEPHONE NUMBER *(Include area code)* +44 (0)1895 616021

FINAL REPORT
CRDF Project N RUM1-1500-ST-04
(1st July, 2007 – 30st September, 2007).

"The distribution function and transport properties of charged particles in strong crossed electric and magnetic fields"

Project director Prof. Dr. A. Ender

INTRODUCTION

For solving the problem to be treated within the project, it was advanced the moment method for solving the Boltzmann equation. Concerning the moment method, the distribution function (DF) is expanded in terms of the Hermit polynomials H_j. Our new fundamental result obtained several years ago forms the basis of the project, i.e., there are a set of the relationships between the matrix elements (ME) of the collision integral [1], [2].

A matrix element $K^j_{j_1,j_2}$ takes a sense of a scalar product of H_j and the collision integral of H_{j_1} and H_{j_2}. Namely, the lack in the MEs of large indices retards any development of the moment method for solving the Boltzmann equation. It was shown that the relationships between the MEs can be used to build up the recurrence procedures to obtain all the MEs if the relatively small number of the linear MEs was known. This is the very value of the new result to be applied in the gas kinetic theory. It should be noted that, in the past, the non-linear MEs were known in a scarce quantity and only with the very small indices and the formulas for the linear MEs (these are easily expressed via the integral brackets) were known only for the small indices, being obtained in a rather general form. The recurrence procedures were developed and built up to calculate the non- linear MEs. Within this calculation, the linear MEs of the large indices were built up to be its integral part. Now, with the MEs to be known, it should be maintained whether or not the moment method will be used to build up the DF. In a course of the project work, the answer "Yes" on this question is obtained. Within a scope of this project, the linear problems were treated as a behaviour of a small ion admixture on the atom background was under investigation. Hence, for solving these problems, it was sufficient the linear MEs to be known. However, an application of the moment method is not determined by the linearity or non-linearity of the collision integral but a DF form, i.e., its deviation from equilibrium. The strong electric field is one of the most serious sources of non-equilibrium (in the case of the charged particles). It

was advanced the non-stationary approach which was successfully used for solving all the problems under investigation in this project. The essence of this approach is in solution of a problem of ion evolution after a sharp switch- on of the electric field. As a result, not only the stationary state is found but the details of the non-stationary process were revealed. The developed method can be used in future when studying the ion behaviour in an oscillated electric field. The problems were solved for the various models of interaction as in a pure electric field as under crossed electric and magnetic fields. In the latter case, a series of interesting phenomena was revealed on the non-stationary stage of the process. All calculations were carried out for the case of equal masses of the ions and atoms. Along with the ion mobility, ion diffusion was studied at the electric field. Here, we studied successfully two-stage non-stationary approach: the DF being established at the given electric field, the non-stationary process was studied after a switch-on of a density gradient which can be directed as along the field (longitudinal diffusion) as transversely to the field (transverse diffusion). Treating the problems inherent for the project, the limits of the moment method application were found. This method turns out to be used in a domain of the moderate and not so strong electric fields. In a region of the very strong electric fields, the DF moments obtain the very large values and DF restoration via its moments can not be accomplished. All that is due to the Grad criterion violation for application of the moment method at the stationary state. We advanced a series of the approaches for solving this problem, i.e., the DF expansion in terms of the spherical harmonics. Here, any knowledge on the MEs with large indices gives a true avenue to build up the kernels of the obtained system of the integro-differential equations.

Our work on the project was proceeded during 11 Quarters, the Quarter 12 being concentrated on this Final Report. Below is the list of the problems considered during our studies on the project. i.e., the study content given quarterly:

Quarter 1. Transfer of the charged particles. Brief literature review. A choice of the way of the study. Matrix elements of the collision integral. New results. The Table of the matrix elements.

Quarter 2. A system of the moment equations in different bases.

Quarter 3. The standard moment method for calculation of the distribution function and ion mobility. Non-stationary processes.

Quarter 4. Study on the standard method convergence for the ion distribution function calculation. CEM–model.

Quarter 5. Numerical solution of non-stationary moment system for various models of interaction.

Quarter 6. Comparison of the ion distribution function under sharp switch-on of the electric field. Details of the moment equation for the charged particles under crossed electric and magnetic fields.

Quarter 7. Non-stationary processes for the charged particles under mutually perpendicular electric and magnetic fields.

Quarter 8. Study on the physical moments of the charged particles under

mutually perpendicular electric and magnetic fields for various models of interaction.

Quarter 9. Evolution of ion distribution function under mutually perpendicular electric and magnetic fields at a sharp switch-on of a weak electric field.

Quarter 10. Evolution of ion distribution function under mutually perpendicular electric and magnetic fields at a sharp switch-on of a moderate electric field for various models of interaction.

Quarter 11. Study on ion diffusion under the electric field.

From this content, one can see that every point which follows is an almost necessary continuation from the preceding ones; it is difficult to change this sequence. At the same time, every Quarterly report is a finite solution of one or another problem. The Final Report is to be logically built up conserving such a sequence when formulating in brief the general results obtaining in the quarters from 1st to 11th.

In the Report Appendix, the additional results obtained for the Customer request are presented. Here, it is given the DF presentation as well as that of ion mobility under the electric field being affected simultaneously by the resonance charge exchange and isotropic scattering.

1 First quarter

First quarterly report [3] consists of three parts.

The first part presents a brief literature review.

There are a lot of studies on mobility of the charged particles. The general results of the investigations accomplished before the 70s of the last century can be found in [4],[5].

Last decades, the most advanced studies concerning mobility of the charged particles have been performed in Australia, K. F. Ness and R. E. Robson being the principal authors of [6, 7, 8, 9]. In Australia, these studies were undertaken on Kumar initiative [10, 11, 12, 13] and continued as a development of his very interesting article [14].

The general result of [14] is in obtaining the direct formulas for calculation of the matrix elements of the collision integral. In a general case, these formulas are very cumbersome and useless for calculation of the matrix elements with large indices because of the multiple sums in the Talmi coefficients. However, they can be applied in the linear ME calculation.

Most of the general results were obtained for the electrons. In this case, the computation problems of matrix elements are easier. From this literature review, a series of conclusions was deduced, the general ones being as follows

1. The moment method is the key when solving the problem which we are confronted unless the unique one.

2. The main problem in the moment method is calculation of matrix elements of collision integral with large indices.

3. When we switch to the weak electric field from the strong one it's appropriate to use the modified moment method. In this method, the temperature of weight maxwellian, about which the DF expansion is undertaken, differs from the temperature of atoms.

4. Potentialities of the standard moment method are far from being exhausted and a domain of its application has not been specified, the method of computation of any arbitrary matrix element being lacking. In the second part, the new results of computation of matrix elements are presented.

The moment method of solving Boltzmann equation is based on the polynomial expansion. We consider the polynomial DF expansion in terms of the spherical Hermite polynomials [1], [16].

$$f(\mathbf{v},\mathbf{r},t) = M(\alpha,c)\sum_j C_j(\mathbf{r},t)H_j(\mathbf{c}). \tag{1}$$

$$H_j = Y^i_{lm}(\Theta,\varphi)c^l S^{(r)}_{l+1/2}(c^2), \quad i=0,1; \quad \mathbf{c}=\sqrt{\alpha(\mathbf{v}-\mathbf{u})}, \quad \alpha=\frac{m}{2kT}. \tag{2}$$

$$Y^0_{lm}(\Theta,\varphi) = P^m_l(\cos\Theta)\cos m\varphi \qquad Y^1_{lm}(\Theta,\varphi) = P^m_l(\cos\Theta)\sin m\varphi \tag{3}$$

Here, an index j consists of four indices (r,l,m,i), M is a Maxwellian distribution with a temperature T and an average velocity $\mathbf{u}$, and $Y^i_{lm}(\theta,\varphi)$ are the real spherical harmonics, $S^{(r)}_{l+1/2}(c^2)$ are the Sonine (Laguerre) polynomials.

The Boltzmann equation is replaced by a system of moment equations:

$$\frac{D(C_i)}{Dt} = \sum_{j,k} K^i_{j,k}C_jC_k, \tag{4}$$

An operator D/Dt in (4) corresponds to the differential (left) side of the Boltzmann equation. When there is not any external force, this operator was studied in [16]. In the presence of external field in the crossed electric and magnetic fields, an operator D/Dt is under study in the later quarters of this project.

Poor progress in the non-linear moment method is due to the large difficulties when calculating the MEs $K^j_{j_1,j_2}$, which are determined as follows:

$$K^i_{j,k} = \int H_i\hat{I}(MH_j, MH_k)\; d^3v/g_i, \quad g_i = \int MH_i^2\; d^3v, \tag{5}$$

Here, $\hat{I}$ is the collision integral, $M(T,u)$ is a Maxwellian, and g_j is a square of a norm of the Hermite polynomial H_j.

The linear matrix elements (LMEs) of the types $K^j_{j_1,0}$ and K^j_{0,j_2} refer to the linear collision integral and could be of the first type $K^j_{j_1,0}$ and second type K^j_{0,j_2}. Further, the LMEs are denoted as Λ. Thus, e.g., in an axially symmetric case we have

$$\Lambda^{(1)}_{r,r_1,l} = K^{r,l}_{r_1,l,0,0}; \quad \Lambda^{(2)}_{r,r_2,l} = K^{r,l}_{0,0,r_2,l} \tag{6}$$

When calculating the kinetics of an ion admixture, we need the LME of the first kind.

In the $1D$ spatial problems, the velocity distribution function is the axially symmetric one, and it's expansion is performed in terms of the spherical Hermite polynomials of two indices (r,l): $H_r^l = S_{l+1/2}^r P_l$.

In [1], it was shown that $3D$ MEs are proportional to the corresponding axially symmetric MEs and the coefficients of proportionality are easily expressed via the Klebsh-Gordan coefficients.

For convergence of (1), the distribution function should satisfy the Grad criterion

$$\int_0^\infty f^2 \exp(c^2) d^3v < \infty. \tag{7}$$

The limits concerning the Grad criterion arise when an expansion is Carried out in terms of the Sonine polynomials but not on the stage of the expansion in terms of the spherical harmonics because the Sonine polynomials do be the orthogonal ones with the Maxwellian weight.

As a base of new relationships between the MEs, we advance the collision integral invariance relative on a set of the basic functions, in terms of which the DF is expanded. The bases distinct in the parameters of the weight Maxwellian are considered. The coefficients of the DF expansion are interrelated via a transfer matrix

$$C_j^1 = \sum_{k=0}^{\infty} D_{j,k}(W_1, W_0) C_k^0. \tag{8}$$

In the axially symmetric case, W incorporates two parameters T and u. Matrix D was built up in [1] with the use of $\alpha - u$ presentation of the Boltzmann equation.

An important theorem results from the invariance principle of the collision integral relative on a choice of the basic functions: the MEs of the collision integral in two different bases are expressed via each other as follows

$$K_{k',j'}^{i'}(W_1) = \sum_i D_{i',i}(W_1, W_0) \sum_{k,j} K_{k,j}^i(W_0) D_{k,k'}(W_0, W_1) D_{j,j'}(W_0, W_1). \tag{9}$$

This expression being differentiating with respect to T_1 or u_1 and setting $W_1 = W_0$ $(u_1 = u_0, T_1 = T_0)$, the following relationships between the matrix elements arise.

"Temperature" recurrence relationships

$$\mu K\binom{a}{a,b}_{r_1,l_1,r_2,l_2}^{r,l} = RK\binom{a}{a,b}_{r_1,l_1,r_2,l_2}^{r,l} + rK\binom{a}{a,b}_{r_1,l_1,r_2,l_2}^{r-1,l}$$

$$-(r_1+1)K\binom{a}{a,b}_{r_1+1,l_1,r_2,l_2}^{r,l} - (r_2+1)K\binom{a}{a,b}_{r_1,l_1,r_2+1,l_2}^{r,l}. \tag{10}$$

Here, $R = r_1 + r_2 - r + (l_1 + l_2 - l)/2$.

In the case of the exponent potentials ($V \sim \frac{1}{r^\nu}$), the MEs are proportional to T^μ, where, e.g., $\mu = 0$ for the Maxwellian molecules, $\mu = 0.5$ for hard spheres, $\mu = -1.5$ in the case of Coulomb interaction.

"Velocity" recurrence relationships

$$
\begin{aligned}
0 = \beta(l-1)K\binom{a}{a,b}^{r,l-1}_{r_1,l_1,r_2,l_2} + \gamma(r-1,l+1)K\binom{a}{a,b}^{r-1,l+1}_{r_1,l_1,r_2,l_2} \\
-\beta(l_1)K\binom{a}{a,b}^{r,l}_{r_1,l_1+1,r_2,l_2} - \gamma(r_1,l_1)K\binom{a}{a,b}^{r,l}_{r_1+1,l_1-1,r_2,l_2} - \\
-\sqrt{(m_b/m_a)}\left(\beta(l_2)K\binom{a}{a,b}^{r,l}_{r_1,l_1,r_2,l_2+1} - \gamma(r_2,l_2)K\binom{a}{a,b}^{r,l}_{r_1,l_1,r_2+1,l_2-1}\right).
\end{aligned}
\tag{11}
$$

$$
\beta(l) = -\frac{l+1}{2l+1}, \quad \gamma(r,l) = \frac{(r+1)l}{2l+1}.
$$

Here, we consider a mixture of particles of two kinds, a and b, with the masses m_a and m_b, correspondingly. The equations are written for the ME of a-kind, scattered by the particles of b-kind. The recurrent equations (10) and (11) hold not only for the matrix elements of the full collision integral but separately for the MEs of the incoming and outgoing terms.

For the ordinary (not oriented) particles, the linear Boltzmann operator meets the Hecke theorem [18], which, in terms of the matrix elements, says: $K^{r,l}_{r_1,l_1,0,0} = 0$ and $K^{r,l}_{0,0,r_2,l_2} = 0$, if $l_1 \neq l$ and $l_2 \neq l$.

Using a recurrence formula (10), (11), it's easy to prove the generalized Hecke theorem.

$$
K^{r,l}_{r_1,l_1,r_2,l_2} \neq 0, \quad if \quad |l_1 - l_2| \leq l \leq l_1 + l_2 \tag{12}
$$

and a parity of l coincides with a parity of $l_1 + l_2$. The recurrence procedures were developed and it was shown that **all the non-linear (and linear) MEs could be found if the linear isotropic (l=0)MEs of the same type are known**. The simple formulas [1], [2] were obtained for the linear matrix elements of the second kind.

A program package which allows to compute all MEs including the MEs with the very large indices for arbitrary masses of colliding particles. Using these programs, the properties of the MEs were studied and some interesting features in their behaviour were found.

New obtained results open new potentialities of the moment method for computation of the DF in a domain of the high energies.

In the third part of report, the Tables of the MEs for different models of interaction were presented. A comparison with the known literature data were made.

2 Second quarter

Second quarter was dedicated to an investigation of a differential operator of the left side of the Boltzmann equation [19].

When solving the spatially non-uniform problems by the moment method, a basic Maxwellian can be chosen as the unique one in all the spatial-temporal points. Such a Maxwellian is name as the global one. Other choice is a local basis, when, for any point, a velocity of the weight Maxwellian coincides with a mean velocity of a gas in this point, and a temperature is determined via a mean thermal energy of the gas. Note, that, besides of the global and local bases, the other ones can be chosen, being the intermediate ones between the formers.

When building up the left side of the moment equations in the global basis, the Boltzmann equation is suitable expressed as follows

$$\hat{A}f = \hat{I}(f, f). \tag{13}$$

Here, $f(\mathbf{v}, \mathbf{r}, t)$ – is the DF, $\hat{I}(f, f)$ – is the collision integral.

$$\hat{A}f \equiv \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m_0} \cdot \frac{\partial f}{\partial \mathbf{v}}, \tag{14}$$

A F is a vector of the external force, a m_0 is a particle mass. An operator $\hat{A}$ is a linear differential operator of the left side of the Boltzmann equation.

In the method of the polynomial expansion in terms of the spherical Hermite polynomials, the DF is presented as (1). The expansion coefficients C_j are expressed via the DF

$$C_j(\mathbf{r}, t) = \frac{1}{g_j} \int f(\mathbf{v}) H_j(\mathbf{c}) d^3v. \tag{15}$$

A normalizing factor g_j is as in [1]

$$g_j = y_{lm}\sigma_{rl}, \quad y_{lm} = \frac{2\pi(1+\delta_{m0})(l+m)!}{(l-m)!(2l+1)}, \quad \sigma_{rl} = \frac{\Gamma(r+l+3/2)}{2\pi^{3/2} r!}. \tag{16}$$

The DF is substituted of its equivalent expression via an infinity vector $\mathbf{C}$ with the components (15). Denote the left-side operator in this expression via $\hat{P}$. Then, we have

$$\mathbf{Q} = \hat{P}\mathbf{C}. \tag{17}$$

The components of a vector $\mathbf{Q}$ are the left sides of the corresponding moment equations. Rewrite (17)

$$Q_j = \sum_{j'} P_{jj'} C_{j'}, \tag{18}$$

The elements $P_{jj'}$ are expressed via the MEs of an operator $\hat{A}$ as follows

$$P_{jj'} = < j|\hat{A}|j' >,$$

where

$$< j|\hat{A}|j' >= \frac{1}{g_j}\int H_j(\mathbf{c})\hat{A}\left(M(\alpha,c)H_{j'}(\mathbf{c})\right)d^3v. \tag{19}$$

Now, we have

$$Q_j = \left(\frac{H_j}{g_j}, \hat{A}f\right) \equiv \frac{1}{g_j}\int H_j(\mathbf{c})\left(\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m_0}\cdot\frac{\partial f}{\partial \mathbf{v}}\right)d^3v. \tag{20}$$

or

$$Q_j = \frac{DC_j}{Dt} + \frac{1}{\alpha^{1/2}}\left(\frac{H_j}{g_j}, \hat{A}_r f\right) + \alpha^{1/2}\left(\frac{H_j}{g_j}, \hat{A}_c f\right), \quad \frac{DC_j}{Dt} = \frac{\partial C_j}{\partial t} + \mathbf{u}\cdot\frac{\partial C_j}{\partial \mathbf{r}}, \tag{21}$$

where

$$\hat{A}_r f = \mathbf{c}\cdot\frac{\partial f}{\partial \mathbf{r}}, \tag{22}$$

$$\hat{A}_c f = \frac{\mathbf{F}}{m_0}\cdot\frac{\partial f}{\partial \mathbf{c}}. \tag{23}$$

A scalar product $(H_j/g_j, \hat{A}_r f)$ can be expressed as

$$\left(\frac{H_j}{g_j}, \hat{A}_r f\right) = \frac{1}{g_j}\int H_j(\mathbf{c})\left(c_x\frac{\partial f}{\partial x} + c_y\frac{\partial f}{\partial y} + c_z\frac{\partial f}{\partial z}\right)d^3v. \tag{24}$$

or

$$\left(\frac{H_j}{g_j}, \hat{A}_r f\right) = \sum_{k=1}^{3}\frac{\partial}{\partial x_k}G_j^{(k)}, \tag{25}$$

$$G_j^{(k)} = \frac{1}{g_j}\int c_k H_j f d^3v. \tag{26}$$

After the cumbersome operations, the expressions which follow are obtained

$$\begin{aligned}
G_j^{(1)} = &-\frac{1}{2(2l+3)}\Big((l+r+3/2)d_{m-1}C^i_{rl+1m-1} - rd_{m-1}C^i_{r-1l+1m-1} \\
&-(l+m+2)(l+m+1)\left((l+r+3/2)C^i_{rl+1m+1} - rC^i_{r-1l+1m+1}\right)\Big) \\
&+\frac{1}{2(2l-1)}\Big(d_{m-1}(C^i_{rl-1m-1} - C^i_{r+1l-1m-1}) \\
&-(l-m)(l-m-1)\left(C^i_{rl-1m+1} - C^i_{r+1l-1m+1}\right)\Big),
\end{aligned} \tag{27}$$

$$\begin{aligned}
G_j^{(2)} = &-\frac{(-1)^{i+1}}{2(2l+3)}\Big((l+r+3/2)d_{m-1}C^{1-i}_{rl+1m-1} - rd_{m-1}C^{1-i}_{r-1l+1m-1} \\
&+(l+m+2)(l+m+1)\left((l+r+3/2)C^{1-i}_{rl+1m+1} - rC^{1-i}_{r-1l+1m+1}\right)\Big) \\
&+\frac{(-1)^{i+1}}{2(2l-1)}\Big(d_{m-1}(C^{1-i}_{rl-1m-1} - C^{1-i}_{r+1l-1m-1}) \\
&+(l-m)(l-m-1)\left(C^{1-i}_{rl-1m+1} - C^{1-i}_{r+1l-1m+1}\right)\Big),
\end{aligned} \tag{28}$$

$$G_j^{(3)} = \frac{l+m+1}{2l+3}\Big((l+r+3/2)C^i_{rl+1m} - rC^i_{r-1l+1m}\Big) + \frac{l-m}{2l-1}\Big(C^i_{rl-1m} - C^i_{r+1l-1m}\Big). \quad (29)$$

Here, we use a denomination

$$d_m = 1 + \delta_{m0} \quad (30)$$

For the terms related with the external force, we have

$$\left(\frac{H_j}{g_j}, \hat{A}_c f\right) = \frac{1}{g_j}\int H_j(\mathbf{c}) \sum_{k=1}^{3} \frac{F_k}{m_0}\frac{\partial f}{\partial c_k} d^3v. \quad (31)$$

Via integration by parts, taking the very fact that the DF turns out to be zero at each infinity limit, (31) can be expressed as follows

$$\left(\frac{H_j}{g_j}, \hat{A}_c f\right) = \sum_{k=1}^{3} \frac{F_k}{m_0} L_j^{(k)}, \quad (32)$$

$$L_j^{(k)} = -\frac{1}{g_j}\int f \frac{\partial}{\partial c_k} H_j d^3v. \quad (33)$$

The following expressions for $L_j^{(1)}$–$L_j^{(3)}$ are obtained

$$L_j^{(1)} = -\frac{1}{2l-1}\Big((l-m)(l-m-1)C^i_{rl-1m+1} - C^i_{rl-1m-1}\Big) - \frac{r}{2l+3}\Big((l+m+1)(l+m+2)C^i_{r-1l+1m+1} - C^i_{r-1l+1m-1}\Big), \quad (34)$$

$$L_j^{(2)} = (-1)^{i+1}\frac{1}{2l-1}\Big((l-m)(l-m-1)C^{1-i}_{rl-1m+1} + C^{1-i}_{rl-1m-1}\Big) + \frac{r}{2l+3}\Big((l+m+1(l+m+2)C^{1-i}_{r-1l+1m+1} + H^{1-i}_{r-1l+1m-1}\Big), \quad (35)$$

$$L_j^{(3)} = \frac{2}{2l+3} r(l+m+1)C^i_{r-1l+1m} - \frac{2}{2l-1}(l-m)C^i_{rl-1m}. \quad (36)$$

Thus, finally, the left side of the moment system with the force terms in the global basis is as follows

$$Q_j = \frac{\partial C_j}{\partial t} + \mathbf{u}\cdot\frac{\partial C_j}{\partial \mathbf{r}} + \frac{1}{\alpha^{1/2}}\sum_{k=1}^{3}\frac{\partial}{\partial x_k} G_j^{(k)} + \alpha^{1/2}\sum_{k=1}^{3}\frac{F_k}{m_0} L_j^{(k)}. \quad (37)$$

The left-side operator of the moment system can be expressed as follows

$$\hat{P} = \frac{D}{Dt} + \frac{1}{\alpha^{1/2}}(\hat{\mathbf{R}}\cdot\frac{d}{d\mathbf{r}}) + \frac{\alpha^{1/2}}{m_0}(\mathbf{F}\cdot\hat{\mathbf{S}}). \quad (38)$$

Any specific form of the MEs of operators $\hat{\mathbf{R}}$ and $\hat{\mathbf{S}}$ is easily found from the formulas for G_j (27)–(29) and L_j (34)–(36), respectively.

When deducing the left side of the moment system in the local basis, a relation of the expansion coefficients the DF C_j in the different bases is used via a transfer matrix D (8).

The basis is featured by a 4-vector W, of which components are the Maxwellian parameters: $\mathbf{u}$ is a mean velocity and T is a temperature.

When using a local basis, W depends on a coordinate and time. When deducing an equation in a point $(\mathbf{r}_0, t_0)$ in the local basis, we take in mind the equations in the global basis with 4-vector W, which coincides with W of the local basis in this point. In a neighbourhood of a point , ($\mathbf{r}_0$, t_0)a relation between the vector $\mathbf{C}$ in the global frame ($\mathbf{C}^{gl}$) and the local one ($\mathbf{C}^{loc}$) is as follows

$$\mathbf{C}^{loc}(W_{loc}) = \hat{D}(W_{loc}, W_{gl})\mathbf{C}^{gl}(W_{gl}). \tag{39}$$

In a point $\mathbf{r}_0, t_0$, we have $W_{loc} = W_{gl}$, and a transfer operator from the global basis to the local one $\hat{D}(W_{loc}, W_{gl})$ turns out to be an identity operator. A linear operator $\hat{P}$ is transformed at the transfer from the global basis to the local one as follows

$$\hat{P}^{loc} = \hat{D}(W_{loc}, W_{gl})\hat{P}^{gl}(\hat{D}(W_{loc}, W_{gl}))^{-1} = \hat{D}(W_{loc}, W_{gl})\hat{P}^{gl}\hat{D}(W_{gl}, W_{loc}). \tag{40}$$

For the left sides of the system of the moment equations we have

$$\mathbf{Q}^{loc} = \hat{P}^{loc}\mathbf{C}^{loc}. \tag{41}$$

An operator $\hat{P}^{gl}$ is a differential operator. So, at $W_{loc} = W_{gl}$, one can replace, in (40), $\hat{D}(W_{loc}, W_{gl})$, at the left, by the identity one at $W_{loc} = W_{gl}$, but $\hat{D}(W_{gl}, W_{loc})$ under a sign of the differential operator can not be excluded. Now, from (41), using (40), we obtain

$$\mathbf{Q}^{loc} = \left(\hat{P}^{gl}\hat{D}(W_{gl}, W_{loc})\mathbf{C}^{loc}\right)|_{W_{loc}=W_{gl}}. \tag{42}$$

As an operator $\hat{P}^{gl}$ incorporates the derivatives with respect to $\mathbf{r}$ and t, and the transfer operator depends on these variables via W_{loc}, now, the spatial-temporal derivatives should be applied to a product $\hat{D}(W_{gl}, W_{loc})\mathbf{C}^{loc}$, and, taking in mind (38), where a $\hat{P}^{gl}$ is determined, we have

$$\mathbf{Q}^{loc} = \hat{P}^{gl}\mathbf{C}^{loc} + \left\{\left(\left(\frac{D}{Dt} + \frac{1}{\alpha_0^{1/2}}(\hat{\mathbf{R}}\cdot\frac{d}{d\mathbf{r}})\right)\hat{D}(W_{gl}, W_{loc})\right)|_{W_{loc}=W_{gl}}\right\}\mathbf{C}^{loc}. \tag{43}$$

In the local basis, the second terms arises in addition. Denote it as $\tilde{Q}$. Take attention to a lack, here, the force terms as, in this part of an operator $\hat{P}^{gl}$, there are not any spatial and temporal derivatives.

After a series of transformations, including a search of the derivatives of a matrix D with respect to $\mathbf{u}$ and T, we obtain

$$\tilde{Q}_j^{loc} = \alpha \frac{DT}{Dt} L_j^{(0)} + \alpha^{1/2} \sum_{k=1}^{3} \frac{Du_k}{Dt} L_j^{(k)}$$
$$+ \sum_{j'} \left(\alpha^{1/2} \frac{\partial T}{\partial \mathbf{r}} \cdot \mathbf{R}_{jj'} L_{j'}^{(0)} - \sum_{k=1}^{3} \frac{\partial u_k}{\partial \mathbf{r}} \cdot \mathbf{R}_{jj'} L_{j'}^{(k)} \right). \quad (44)$$

In the axially symmetric case, $i = m = 0$, and there is only unique component of a vector $\mathbf{u}$, denoted by u. An equation (44) takes a form

$$\tilde{Q}_j^{loc} = \alpha \frac{D\tilde{T}}{Dt} L_j^{(0)} + \alpha^{1/2} \frac{Du}{Dt} L_j^{(3)}$$
$$+ \alpha^{1/2} \frac{\partial \tilde{T}}{\partial z} \sum_{j'} R_{jj'}^z L_{j'}^{(0)} - \frac{\partial u}{\partial z} \sum_{j'} R_{jj'}^z L_{j'}^{(3)}. \quad (45)$$

Here,

$$L_j^{(0)} = (r + \frac{l}{2}) C_{rl}^{(loc)} - r C_{r-1l}^{(loc)},$$
$$L_j^{(3)} = \frac{2}{2l+3} r(l+1) C_{r-1l+1}^{(loc)} - \frac{2}{2l-1} l C_{rl-1}^{(loc)},$$
$$\sum_{j'} R_{jj'}^z L_{j'}^{(0,3)} = \frac{l+1}{2l+3} \left((l + r + 3/2) L_{rl+1}^{(0,3)} - r L_{r-1l+1}^{(0,3)} \right)$$
$$+ \frac{l}{2l-1} \left(L_{rl-1}^{(0,3)} - L_{r+1l-1}^{(0,3)} \right). \quad (46)$$

We verified that the continuity, motion and energy equations in a form advanced in [20] are results from several first moment equations in the local basis.

3 Third quarter

In Quarter 3 [21], the research ways to be developed during several next quarters were formulated concerning ion behaviour under the external electric field.

The project as a whole is devoted to computation of the transport coefficients and the DF of an ion admixture on an equilibrium (Maxwellian) gas background via the moment method either under the external electric field or crossed electric and magnetic fields.

In those problems, the collision integral is linear, and, in the moment method, it corresponds to the linear MEs of the first kind. Note, that the linearity of the collision integral does not exclude a strong ion non-equilibrium. Under an increase in the electric field strength, a deviation from the Maxwellian distribution could be very large. Hence, the general problems of the moment method convergence can be

studied with these (linear) problems as an example. When working on this project, we consider the ions of which mass coincides with a mass of atoms. We denote the standard moment method as that in which the distribution function expansion is carried out about the Maxwellian with a temperature of atoms. Along with that, an expansion can be performed about other Maxwellian of which parameters do not coincide with the parameters of the background atoms distribution; such approach is denoted as the modified moment method. Under moderate fields, the standard moment method was consider by Kihara [22]. He developed an iteration method to be applied with the minor modifications in [4]. Any knowledge on the elements of the collision integral is urgently needed to succeed in the moment method application. With an increase in the electric field strength, the number of the MEs needed for the iteration process convergence. In [4], the formulas for the ME calculation at no so large indices were tabulated. The authors emphasized that this collection of the formulas is the most general one, incorporating the Ferziger, Caper results [23] as well as the authors themselves. Overall, there are the formulas for calculation of about two tens of the MEs in these Tables.

We compared the MEs calculated via the formulas of [4] with our results shown in the Tables in the first Quarterly report. It turns out that there is a series of distinctions in the ME definitions: opposite sign in the collision integral, different definition of the expansion coefficients of the DF and different choice in of a measurement unit of collision frequency. Further comparison results in an ultimate coincidence of our data with calculation via the Tables from [4] except a single ME because of, eventually, with misprint in the corresponding formula in [4], taking in mind all the distinctions mentioned above.

To perform the calculations, a transfer to the dimensionless variables was carried out. As a velocity unit it was chosen always the thermal velocity of atoms.

$$v_T \equiv \sqrt{\frac{1}{\alpha}} \equiv \sqrt{\frac{m}{2kT}} \tag{47}$$

As a time unit, that of between collisions, τ, was chosen. If a scattering cross section $\sim 1/g\lambda$, where g is a relative velocity of the colliding particles, τ does not depend on a velocity; a constant frequency between collisions (a time between collisions) is inherent for the Maxwellian and similar to the Maxwellian (pseudo-Maxwellian molecules; BGK–model). In these cases, a choice in time unit is clear, and, when transferring to these units, for dimensionless electric field, we obtain

$$\epsilon = \frac{eE\tau v_T}{m} \equiv \frac{eE\tau}{\sqrt{2kTm}} \tag{48}$$

However, for other interaction cross sections, the collision frequency depends on a relative velocity, and a choice in time unit is not so clear. E.g., for hard-sphere model, the scattering cross section does not depend on a velocity, and it turns out that not the collision frequency is constant but a free path length λ.

$$\lambda = N_a \pi d^2, \tag{49}$$

Where N_a is the concentration of atoms, and d is a diameter (in general case, a sum of radii of colliding particles).

The MEs are calculated, also, at the definite normalization, and always the unique question arises what to be included in a definition of collision frequency and what should be incorporated in the ME. In [4], the collision frequency for hard spheres of equal masses is chosen as follows

$$\frac{1}{\tau_M} = \lambda \left(\frac{4kT}{\pi m}\right)^{1/2}. \tag{50}$$

When calculating the dimensionless MEs, we choice the collision frequency as follows

$$\frac{1}{\tau_E} = \lambda \left(\frac{kT}{m}\right)^{1/2}. \tag{51}$$

In these very units, in the first quarter, the ME Tables were presented. In the third quarter, as the time unit, it was advanced τ_E, when transferring to the dimensionless Boltzmann equation, and such normalization was named the standard one. In these units, for the dimensionless electric field, we have

$$\epsilon = \frac{eE\lambda}{\sqrt{2}kT}. \tag{52}$$

Further, beginning in the sixth quarter, a definition of the standard normalization was changed; conserving the velocity unit (47), a time unit was changed. For the interaction models with a constant free path length, naturally, λ should be chosen as a length unit. With the chosen units of length and velocity, τ is determined uniquely as $\tau = \lambda/v_T$. Thus, for the particles with the velocity independent scattering cross section, the standard normalization was named that, for which

$$\frac{1}{\tau} = \lambda \left(\frac{2kT}{m}\right)^{1/2}. \tag{53}$$

When transferring to the final standard normalization, we need to denominate again all the dimensionless MEs, namely, to divide by $\sqrt{2}$ for the models with $\lambda = const$.

For the final standard normalization for ϵ we have

$$\epsilon = \frac{eE\lambda}{2kT}. \tag{54}$$

As it was noted earlier, all calculations beginning from the sixth quarters are presented in the final standard normalization. However, in the fifth quarter, for the models with $\lambda = const$, in the results involving the standard normalization, one should denominate all the values of ϵ and t, namely, ϵ is to be divide by $\sqrt{2}$, and a time is to be multiplied by $\sqrt{2}$, referring to the Figure captions as well as in text.

Go to the dimensionless variables. As a velocity unit, we choice a thermal velocity v_T and denote a dimensionless velocity $\mathbf{c}$

$$\mathbf{c} = \frac{\mathbf{v}}{v_T}, \tag{55}$$

and, for the dimensionless DF and time, we conserve the previous denominations: $f(\mathbf{c}, t)$, and t. In the dimensionless form, the Boltzmann equation takes a form

$$\frac{\partial f}{\partial t} + \epsilon \frac{\partial f}{\partial c_z} = \hat{J}(f, M), \tag{56}$$

where $\hat{J}(f, M)$ is the dimensionless collision integral. In an equation (56), the z-axis direction coincides with the electric field direction. The corresponding system of the dimentionless moment equations takes a form

$$\frac{\partial C_{r,l}}{\partial t} + \epsilon \Big(\frac{2}{2l+3} r(l+1) C_{r-1,l+1} - \frac{2l}{2l-1} C_{r,l-1} \Big) = \sum_{r_1} \Lambda_{r,r_1,l} C_{r_1,1} \tag{57}$$

Here, if the terms with a temporal derivative are omitted and the distinctions in the definitions of the collision frequency as well as normalizations are taken into consideration, the system coincides as whole with the stationary system in [4], for which solution, Kihara [22] advanced the iteration method.

As a rule, when solving the kinetic equations for the ions at the external field, the stationary state of a system is looked for, i.e., the stationary values of mobility and the stationary DF. Namely, in the third quarter, we advanced the non-stationary approach to find the stationary solutions. For brevity, we call such approach the **non-stationary moment method**. In this method, the full system of the moment equation is used, including the terms with a temporal derivative, and it is supposed that, till $t = 0$, the system is in a total equilibrium, i.e., the ions has the Maxwellian distribution coinciding with the atom one. At $t = 0$, the electric field is instantly switched on. By solving the non-stationary system of the moment equation, the transient process is studied, beginning with the electric field switch-on and terminating when obtaining the stationary state. As a result, we find the ion mobility and DF at the stationary state with no use of the complicated iteration procedure. Besides, the ion behaviour in a course of the transient process is of interest itself.

The non-stationary approach was used successfully during all the work on this project including the study on the effect of the crossed electric and magnetic fields on the ions as well as the ion diffusion under the electric field.

Clearly, to solve via the moment method the problems formulated above, the number the involved moment equations should be essentially larger as compared with the stationary mobility calculation.

Development of an algorithm to solve the non-stationary problem as a perspective gives an opportunity to go to an important applied problem concerning the ion behaviour under alternate electric field.

When solving the non-stationary moment system, the 4th-order Runge-Kutta method is used.

When studying the moment method convergence, it is of importance to have an analytical solution of the problem under consideration, at least, in several particular cases. Such an interaction cross section was obtained with which an analytical solution not only of the stationary problem but the non-stationary process could be built up.

Under consideration is a model in which a cross section is inversely proportional to the relative velocity, and the angulat part takes a form $\delta(\theta - \pi)$. This stationary model of the charge exchange with the Maxwellian dependence of the cross section on the relative velocity we call the CEM–model. As the collision frequency in this model does not depend on the velocity, it is eliminated from the collision integral; the outgoing term of the collision integral equals $-f(\mathbf{v})/\tau)$ where $f(\mathbf{v})$ is the ion DF. In it's turn, δ-shaped angular dependence results in proportionality of the incoming term to the atom DF. With a small ion admixture on the atom background, the atom DF is in equilibrium, i.e., it takes a form of the Maxwellian distribution. As a result, for CEM– model, a collision operator takes a form

$$\hat{I} = \frac{M(c) - f(\mathbf{c})}{\tau}. \tag{58}$$

In it's form, it coincides with the linear BGK–model [24], which is frequently used to obtain the analytical and numerical solutions in a gas of unique sort. If the BGK–model is made with aim to simplify the collision integral and has not the physical ground, on contrary, the CEM–model is an implication of a choice of a well defined scattering cross section. Simultaneously, it has all the gains of BGK–model and can be used to build up the analytical solutions. Present with no derivation the general formulas describing the analytical solutions. At the chosen measurement units, a drift velocity of ions equals

$$v_d = C(0,1)/2,$$

where

$$C(0,1) = 2\epsilon(1 - exp(-t)). \tag{59}$$

Thus, in the chosen measurement units, the mobility goes to unity at $t \to \infty$.

The stationary solution for the DF takes a form:

$$f(c_\rho, c_z) = \frac{\sqrt{\pi}}{2} M(c_\rho) M(c_z) \frac{1}{\epsilon} \exp\left(c_z - \frac{1}{2\epsilon}\right)^2 \left(1 + erf(c_z - \frac{1}{2\epsilon})\right). \tag{60}$$

The non-stationary solution for the DF at the axis of symmetry takes a form:

$$f(t, c_z) = \frac{\sqrt{\pi}}{2} M(c_z) e^{(c_z - 1/(2\epsilon))^2} \frac{1}{\epsilon} \Big(erf(\epsilon t - (c_z - \frac{1}{2\epsilon})) + \\ + erf(c_z - \frac{1}{2\epsilon})\Big) + e^{-t} M(c_z - \epsilon t). \tag{61}$$

To obtain these solutions, the method of characteristics was used. Note, that, for CEM–model, the solution can be built up, also, for other non- stationary processes, e.g., to describe the ion behaviour under periodical external electric field. However, a solution of such a problem is far from the scope of this report.

As an example, Fig. 1 shows the calculation results via the analytical formulas of the DF at $\epsilon = 2$. Figure 1a is the DF dependence on C_z at the axis of symmetry ($c_\rho = 0$) at the different times. Figure 1b is an analogous dependence referred to the stationary DF. Here, it is introduced a denomination $f_r(0, c_z) = f(0, c_z)/f_{stac}(0, c_z)$, where $f_{stac}(0, c_z)$ is the DF at the axis of symmetry at the stationary state. At $t \to \infty$ $f_r \to 1$. In a velocity space, along the electric field, one can see the motion of a front of which behind the stationary DF arises.

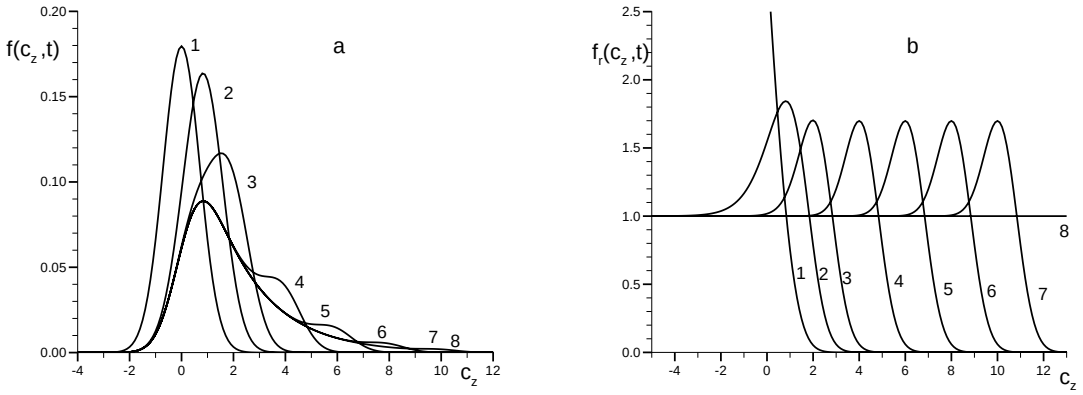

Figure 1: Distribution function $f(c_z, t)$ at the axis of symmetry at the different times after a switch-on of the the electric field strength of $\epsilon = 2$ (a) and the distribution function referred to the stationary DF $f_r(c_z, t) = f(c_z, t)/f(c_z, \infty)$(b). The times $t = 0$ correspond to curve (1), 0.5 (2), 1 (3), 2 (4), 3 (5), 4 (6), 5 (7), 10 (8).

As it can be seen in Figure, relaxation in a domain of the large positive values of c_z proceeds exclusively slow, and the stationary solution yielded in this velocity domain developed with a large delay.

When studying the moment method convergence, the Grad criterion feasibility is of crucial value, for which aim a sufficiently rapid decrease in the DF at the infinity is needed. It is shown that the stationary solution for the interaction model under consideration does not satisfy the Grad criterion. However, at any finite time t in a course of the non-stationary process, the DF decreases in a large velocity domain in a such rapid manner that the Grad criterion holds and there are not any general limits to the moment method convergence. As a result, in a course of the non-stationary process, a large part of the stationary DF can be built up. Besides of the analytical solution for the DF in the case of CEM–model, we obtain successfully the analytical solution for the moment of the DF.

$$C_{r,l}(t) = B_{r,l}\epsilon^N S_N(t), \quad N = l + 2r. \tag{62}$$

Here, $B_{r,l}$ is a coefficient independent on either time or ϵ. It equals

$$B_{r,l} = (-1)^r \frac{(2l+4r)!!(2l+1)}{2^r(2r+2l+1)!!}. \tag{63}$$

A function $S_N(t)$ takes a form

$$S_N(t) = 1 - \exp(-t) \sum_{k=0}^{N-1} \frac{t^k}{k!}, \tag{64}$$

and it is the same for total layer $N = l + 2r$. This function turns out to be zero at $t = 0$ and 1 at $t \to \infty$.

4 Fourth quarter

Quarter 4 deals with a study of the moment method convergence involving the CEM–model [25].

In Quarter 3 for CEM–model it was obtained the analytical solution describing the non-stationary process after an instant switch-on of the electric field. When obtaining such solutions, the interesting perspectives are opened for the convergence search. Actually, for CEM–model one can always compare the DF restored via the moments with the exact solution and, hence, always, an opportunity occurs when the moment method converges. The conclusions obtained for CEM–model turn out to be true for other interaction potentials of ions and atoms as well, a problem concerning the DF convergence to be restored via the moment, analytically built up, being separated from the study of the errors which arise during the numerical solution of the moment system itself.

The main reason of the moment method divergence is the Grad criterion violation (7) under the strong deviation from equilibrium. A slow convergence takes place under rather strong deviations from equilibrium when the system is the nearest to a state in which the Grad criterion does not hold. It was shown analytically [21] that the DF at the stationary state ($t \to \infty$) does not satisfy the Grad criterion even at a low value of the electric field strength. However, in a course of the non-stationary process (at finite t), the DF satisfies the Grad criterion. Clearly, that at a long time, when the DF differs little from the stationary DF, which does not satisfy the Grad criterion, the number of the moments necessary for convergence should be very increasing. Any question concerning the true state of art can be answered only with the relevant calculation.

When solving numerically the kinetic problems, the cut-off of the system of the moment equations is necessary to be carried out, i.e., instead of an infinite system of the moment equations the finite one is treated. Cutting-off the moment system, certain domain to which the indices l- and r are attributed is separated out. Under investigation is to be the effect of the form of such a domain. The aim is to obtain

such an increment of the moment located on the domain boundary being dependent on only the moments located in its interior. In this case, the substitution of zero values for the moments in its exterior has not any influence on the moment increment within this domain.

A comparison between the calculations with the different domain of summation was carried out. Triangular domain $REG1$

$$l \leq N_0 - r \tag{65}$$

Triangular domain $REG2$

$$r \leq (N_0 - l)/2. \tag{66}$$

and a rectangular domain $REG3$

$$r < R_0, \qquad l < L_0 \tag{67}$$

Here, R_0 is the number of the Sonine polynomials, L_0 is the number of the Legendre polynomials. For CEM–model, the most natural is a domain $REG2$. The calculation results are not inferior than those with a choice of a domain $REG1$ with the same amount of the moments. The calculations dealing with the triangular domain are more optimal than with the rectangular one. Nevertheless, to exclude the very large values of l, i.e., with their aid to decrease further the number of the moments, the summation can be carried out over a trapezoidal domain $REG4$

$$l \leq N_0 - r, \quad l \leq L_0, \quad L_0 \leq N_0 \tag{68}$$

With the relevant choice of N_0 and L_0. Further calculations involved in the project, as a rule, were carried out with a summation over a domain $REG4$.

Usually, the moment method is applied to calculate the transport coefficient.

The solution of the system of the moment equations with the comparatively large number of the moments is needed, along with [4], because of two reasons: first, the larger number of the Sonine polynomials is needed to be considered as compared with the common treatment performed via the simplest variant of the Chapman-Enskog method, second, when increasing the electric field strength, the senior moments can affect the junior ones via those the needed transport coefficients are expressed.

We are confronted with even the most ambitious problem, i.e., to build up the DF with the moment method. Moreover, we propose to solve this problem in the non- stationary statement. With the success obtained, the transport coefficients will be found and their temporal dependences will be built up. Clearly, the calculation of the transport coefficients of a domain in the plane (l, r) only will be of essentially fewer than that for the DF calculation.

To build up the DF via the moments, first, $f_l(c)$ (DF in l-subspace) are calculated) via a formula

$$f_l(c) = M(c) \sum_r c^l S^r_{l+1/2}(c^2) C_{r,l} \tag{69}$$

Then, via a formula

$$f(\mathbf{c}) = \sum f_l(c) P_l(\cos(\theta)) \tag{70}$$

the DF is found.

The principal studies are performed at the axis of symmetry. Consequently, the numerical studies were performed for the cases of the weak, moderate and strong electric field. These regions are determined as follows:
Weak electric field – $\epsilon \leq 0.1$,
Moderate electric field – $0.1 < \epsilon < 1$,
Strong electric field – $\epsilon \geq 1$.

Using the analytical formulas for the moments, it was shown that, for CEM–model, the moment method results in a divergent series in attempting to build up the DF at any l and any value of the electric field ϵ.

However, at any finite time, the absolute values of the moments, i.e.,$|C_{r,l}|$) decrease with an increase in r at any l, and the DF expansion in terms of the moments should principally be convergent.

When investigating the weak electric field, the problem of the DF building-up in the stationary state receives primary emphasize. It turns out that, in this region of ϵ, there is the minimum on the $|C_{r,l}|(r)$ dependences. Even on the upper boundary of this region of $\epsilon = 0.1$, the minimum is very deep, at the minimum, $|C_{r,0}|$) is 12 orders lesser than $|C_{0,0}|$; the minimum coordinate $r = 25$.

At the small values of ϵ, a departure from equilibrium is small. Hence, the number of the moments needed to build up the DF should be small. It results in a real possibility to build up the stationary DF via the moments involving the true choice of the cut-off domain. As one would expect, the cut-off should be performed somewhere within the minimum neighbourhood.

When transferring in a domain of the moderate field, with an increase in ϵ, there is a very fast decrease in a minimum depth on a $|C_{r,l}|(r)$ dependence. An attempt to build up the stationary DT via the stationary moment does not result in a satisfactory yield already at $\epsilon = 0.25$. In the iteration method, advanced by Kihara [22], to built up the stationary solution, the moments need to be found. Emphasize, that, for CEM–model, the stationary moments are known. Nevertheless, there is not a grain of hope to succeed in building-up the stationary DF in a domain of the moderate field via the iteration method. This does not ruled out the possibility to find several moments with the small indices, particularly, a drift velocity with which the mobility can be found using this method.

Quite to the contrary is a deal concerning the DF restoration via the moments in a course of non-stationary process. It is due to a form of $|C_{r,l}|(r)$ dependences. In any finite time, beginning with certain r, the moments decrease. First, there is observed the minimum at $|C_{r,l}|(r)$ dependence, then, there is the maximum. When increasing t, the coordinate of the maximum and its height increase with no limit. At the maximum left, there is a region r, in which interior a dependence $C_{r,l}(r)$, corresponding to the non-stationary solution, coincides with the stationary one, and this region is enlarged with an increase in t.

For the convergence of a series (69) to the exact solution, one needs to pass the maximum, when summing over r, and enter a region r where the values of $|C_{r,l}|$ are sufficiently small. At very large time when the coordinates of the maximum are large, such a summation turns out to be very complicated. However, in the domain of the moderate electric field, the DF restoration is successfully performed up to sufficiently long time.

In [21], it was shown that the process to maintain the stationary state is accompanied by propagation of a front along c_z-axis in the velocity space. Behind this front, the stationary state develops almost immediately. In a domain of the negative velocities, the stationary state yielding is over approximately by $t = 1$. By the sufficiently long time, the DF coincides with the stationary one within a very wide region c_z.

As a result, with the aid of "non-stationary moment method", we succeed in solving the seemingly unsolvable problem to build up the stationary solution in a wide region of c_z.

In a domain of the strong electric field, the problem of passing the maximum turns out to be more complicated. For the convergence, the cut-off region should be enlarged strongly in a (r, l)-plane.

When calculating $f_l(c)$, there is a product of the Sonine polynomials $S^r_{l+1/2}(c^2)$ and the moments $C_{r,l}$ under a sign of a sum (69). The moments have the alternating sign with a change in r, this is true for the Sonine polynomials as well. As a result, the terms of the sum change their signs and the sum turn out to be many orders lesser that the absolute value of a maximum term of the sum. If, when passing the maximum, a ratio of the maximum general term to the exact value is greater than 10^{16}, the loss in accuracy occurs and the true value is out of reach, the number of the terms of the sum being made as large as is wished.

It was maintained that, if ϵ and t are simultaneously changed but in the manner of a product $\epsilon \cdot t$ being constant, the dependences $|C_{r,l}|(r)$ turn out to be very similar. Thus, certain parameter of similarity is found – $\epsilon \cdot t$. It is shown that a good convergence takes place at $\epsilon \cdot t < 5$.

In a domain of the strong electric field, the "non-stationary moment method" to build up the stationary regime is improved.

The analytical solutions for CEM–model, being built up, simultaneously, both for the Boltzmann equation itself and the moment system, gave an insight into capabilities of the standard moment method.

In Fig. 2, for CEM–model, the DF at the axis of symmetry ($c_\rho = 0$) is shown for different times at $\epsilon = 2$. It shows in what manner increases the number of the Sonine R_0 and Legendre L_0 polynomials to be taken into consideration in the DF expansion.

For $c_z > -4$, there is a fine coincidence with the corresponding analytical solutions. For $c_z < -4$, a very fast yield toward the stationary solution but, as a time increases, the more difficulties arise when building up the solution via the moment method.

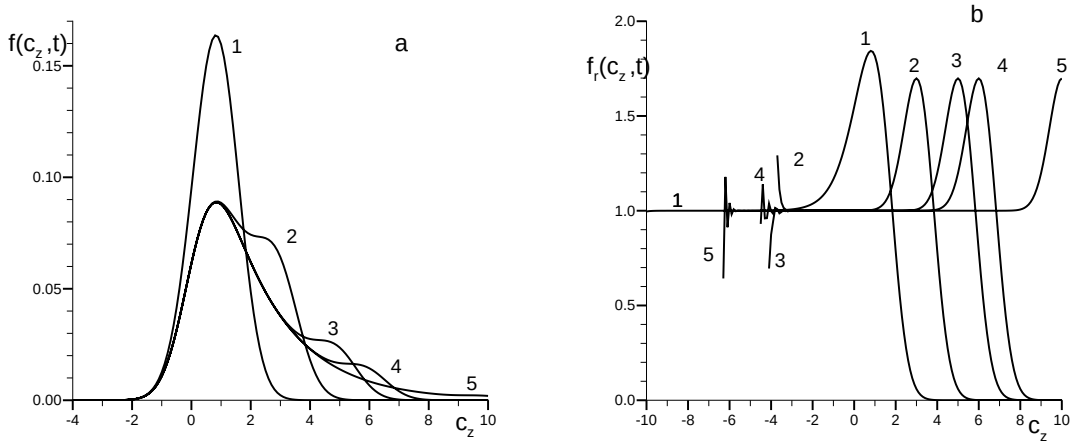

Figure 2: The distribution function at the symmetry axis at different times for CEM–model at $\epsilon = 2.0$.**a**– $f(c_z, t)$. **b** – $f(c_z, t)/f(c_z, \infty)$. Curves 1-5 refer to a time $t = 0.5$ ($L_0 = 35$, $R_0 = 23$); 1.5 ($L_0 = 30$, $R_0 = 26$); 2.5 ($L_0 = 37$, $R_0 = 70$); 3 ($L_0 = 75$, $R_0 = 100$); 5 ($L_0 = 100$, $R_0 = 280$).

Thus, with an increase in the electric field, the DF building-up via the moment method is limited by the shorter times, and, to solve the problems at the very strong fields, it is necessary to develop the new methods in kinetic problems solution. In spite of the above objection, basing on the studies performed, the conclusions can be derived as follows: 1. The non-stationary moment method to build up the DF is developed. This method is an advance in the standard moment method involving the DF expansion about the Maxwellian of which temperature coincides with the temperature of the background atoms. 2. Using the non-stationary moment method , we succeed in building up the solution in a domain of the moderate and strong electric fields. 3. At least, by one order of value is increased ϵ for which, using the moment method, the DF turns out to be build up at the stationary state.

5 Fifth quarter

In Quarter 5 [26], the solutions of the non-stationary problem were obtained under a sharp switch-on of the electric field for the ions with several interaction models. In the detailed 5th quarterly report, there are many detailed proofs and research results as well as many illustrations. The Quarter 5 being terminated, this detailed report was sent to EOARD as well as the brief report.

In the first section of the 5th quarterly report [26], with an example of CEM–model, the studies on the moment method convergence were continued.

In [25], it was revealed that, in the case CEM–model, an increase in the electric field ϵ as well as a time t result in poorer convergence of the DF restoration via the moments. It is because of the high maximum arising on a dependence of

the moments $|C_{rl}|$ on r at a fixed l; it's height and it's r-coordinate increase in time. When calculating the DF, one should go essentially to the right of this maximum, The method of the MEs calculation of large indices (matrix elements of the collision integral) developed by us is ready to make it. However, if the degree of non-equilibrium is high and the moment values of the maximum are enormous, it results in a loss in accuracy of the DF calculation during summation as well as in the convergence default. It occurs when the DF is slightly distinct from the stationary one which does not satisfy the Grad criterion. In the 5th quarterly report, the study of the attempts to ameliorate the moment method convergence due to a change in basis, i.e., an increase in the temperature of the weight Maxwellian, were studied. Earlier, the need of the basis change was mentioned in [5], [9], [10]. However, in these works, the ME building-up in the new basis was performed via the calculation of the complicated integrals with the large time consumption [9] with no certainty in a good accuracy, the ME building-up being successfully carried out with the rather small indices ($l \leq 7$, $r \leq 10$) only.

When investigating the matrix elements of the collision integral in [1], a point on a transfer from one basis to another was considered in details. The transfer matrix (operator $\hat{D}$) was built up, the bases being distinct, in the general case, not only in the temperature but the mean velocity of the Maxwellian. When building up an operator $\hat{D}$, we use α-u-representation of the Boltzmann equation [17], [27]. In the report, the detailed derivation of a matrix D, which relates both bases, which are distinct only in their temperature. With the matrix D, one can obtain the ME in the new basis via the ME in the initial basis. Using the results of the 2nd quarter, we can pass from the global basis to the local one and solve the moment system in the new basis; we call such approach **the modified moment method**. In general, the parameters of the weight Maxwellian experience the changes all the time in a course of the non- stationary process. To verify the capabilities of this method, there is no need in solving the system of the moment equations in the new basis with the temporal temperature dependence. One can transform the moments, which were built up in an initial basis C^0_{rl}, to the new basis and obtain C^1_{rl}.

Thus, when transferring from a basis T_0 to the T_1 one, the moments change as follows

$$C^1_{rl} = \sum_{k=0}^{\infty} D_{rl,kl}(\tilde{T}_1, \tilde{T}_0) C^0_{kl}, \tag{71}$$

It was considered an example: $\epsilon = 1$, $t = 5$. With such a field and time, the moments C_{rl} in a basis T_0 are very large.

It was developed a special code in which the moments built up in a basis T_0 (Bas0) were transformed via a formula (71) to a basis T_1 (Bas1). The relevant parameter for this transformation is a ratio $RT = T_1/T_0$.

In Fig. 3, the calculation results at $RT = 1.5$ are presented.

One can see in this figure that, in a subspace $l = 0$, a transfer to the new basis is accompanied with a change in the C_{rl} maximum by ~ 5 orders of value, in a subspace $l = 10$, by ~ 3 orders. In both subspaces, there is a shift of the distribution C_{rl}

toward the domain of small r – a coordinate of the maximum is strongly decreases, i.e., at $l = 0$, it changes from 24 to 9. Thus, even a comparatively small change in the basis temperature results in an essential facilitation in the DF restoration via its moments.

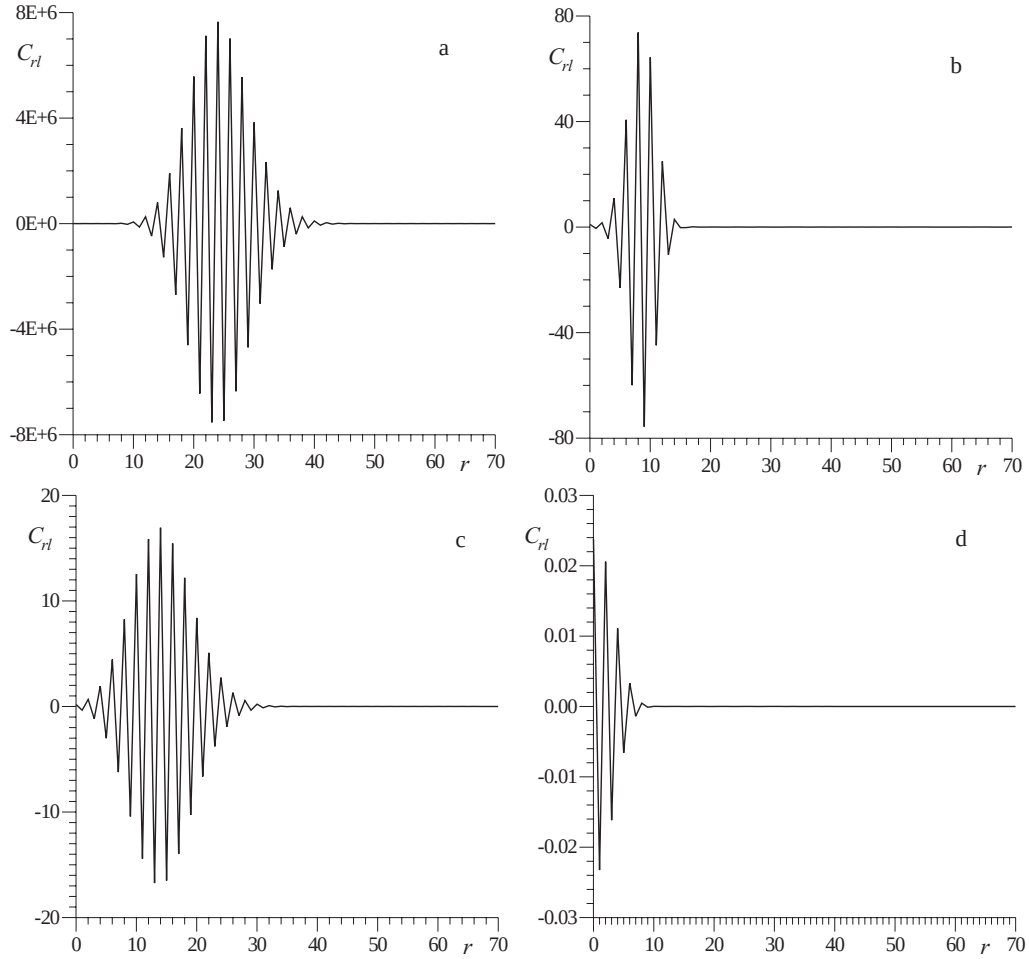

Figure 3: The dependence of the moments C_{rl} on r for CEM–model in two bases with the temperatures T_0 and T_1; $RT = 1.5$, $L_0 = 37$, $R_0 = 70$. Figure a – Bas0, $l = 0$, b – Bas1, $l = 0$, c – Bas0, $l = 10$, d – Bas1, $l = 10$.

To build up the DF with the same accuracy as for the initial basis, it is needed essentially smaller number of the moments. The results obtained lend support to the utility in involving the modified moment method.

Our experience in the moment system transformation from one basis to another gives the way to solve this problem by the efforts of our collective in the case of a new project devoted, generally, to the development of the modified moment method.

In the 5th quarter, also, the formulas relating the physical moments and the moments C_{rl} were obtained. Beside of such moments as a drift velocity u_d and

mobility $K = u_d/\epsilon$, the expressions for a total energy, total energy flow, thermal energy, and heat flow.

In the axially symmetric case, an expansion of the angular dependence of the DF is carried out only in terms of the Legendre polynomials.

The first physical moment, i.e., concentration coincides with C_{00}. In the process under consideration, n experiences no changes, and we set $C_{00} = 1$, i.e., the DF is said to be normalized by unity. The physical moment are the mean values of the physical quantities which are expressed, usually, via the different combinations of the various powers of the particle velocities. Mean values to be normalized by unity are determined as follows

$$\langle c_x^i c_y^j c_z^k \rangle = \int f c_x^i c_y^j c_z^k d^3c. \tag{72}$$

A mean velocity of the ions is their drift velocity to be denoted by u_d. For it we have

$$u_d = \langle c_z \rangle = \frac{1}{2} C_{01} \tag{73}$$

Determine the mean energy of the ions $W = \langle mv^2/2 \rangle$ in a frame related to the atoms.

For dimesionless energy we have

$$\frac{W}{kT} = 3/2(C_{00} - C_{10}) \tag{74}$$

The mean dimensionless energy flow in the frame is expressed via C_{rl} as follows

$$\frac{Q}{kT\sqrt{2kT/m}} = 5/4(C_{01} - C_{11}) \tag{75}$$

To determine the mean thermal energy, mean heat flow and other moments related with a chaotic motion of the particles, it is convenient to involve a "proper" velocity of the particles

$$\mathbf{C} = \mathbf{c} - \langle \mathbf{c} \rangle = \mathbf{c} - \mathbf{u}_d \tag{76}$$

The ions thermal energy — the energy in a frame related to the ions is determined as follows

$$\frac{W_T}{kT} = \langle C^2 \rangle, \tag{77}$$

and a heat flow

$$Q_T/(kT\sqrt{2kT/m}) = \langle C^2 C_z \rangle \tag{78}$$

After a series of transformations we obtain

$$\frac{W_T}{kT} = 3/2(C_{00} - C_{10}) - 1/4C_{01}^2, \tag{79}$$

$$Q_T/(kT\sqrt{2kT/m}) = -5/4C_{11} + 5/4C_{10}C_{01} + 1/4C_{01}^3 - 1/2C_{02}C_{01} \tag{80}$$

In addition, determine other physical moments from the DF. A chaotic energy in the longitudinal direction, i.e., along the electric field

$$\frac{W_T^l}{kT} = \langle {C_z}^2 \rangle, \tag{81}$$

and the transversal energy

$$\frac{W_T^t}{kT} = \frac{W_T}{kT} - \frac{W_T^l}{kT} \tag{82}$$

For them we have

$$\frac{W_T^l}{kT} = 1/2(C_{02} + C_{00} - C_{10}) - 1/4C_{01}^2, \tag{83}$$

$$\frac{W_T^t}{kT} = C_{00} - C_{10} - 1/2C_{02} \tag{84}$$

The physical moments behaviour is calculated after a sharp switch-on of the electric field for several interaction models:

Pseudo-Maxwellian molecules, a total cross section is inversely proportional to the relative velocity $\Sigma = C/g$, angular part of the cross section is isotropic

CEM–model, the total cross section is also inversely proportional to the relative velocity, angular part of the cross section — the resonance charge exchange refers to 180°- scattering.

CEHS–model, the total cross section does not depend on a velocity $\Sigma = const$, angular part of the cross section is the same as for CEM–model of the charge exchange.

HS–model, hard sphere model, the total cross section does not depend on a velocity $\Sigma = const$, angular part of the cross section – isotropic scattering.

Such normalization is advanced that, in the limit of very weak fields, the mobility equals unity. Such normalization is named, further, **non-standard normalization**.

It was shown that, with the diagonal interaction matrix (CEM–model and pseudo-Maxwellian molecules), the temporal dependence of the ion mobility after the field switch-on is the universal curve and does not depend on ϵ as well as the limit (stationary) value of mobility. For other two models this dependence changes with an increase in ϵ. The limit (stationary) value of mobility depends, also, on the electric field, i.e., it decreases with an increase in ϵ.

The transformation to the non-standard normalization occurs as follows: the problem is solved using the standard normalization at very small ϵ, the mobility is found, being, in general, non unity, this value is a coefficient by which all the MEs are divided, Clearly, that, changing the MEs, we reform the coefficient in the collision integral, i.e., we change the time unit τ. As a result, both the time scale and the value of the dimensionless electric field ϵ change. After the transfer to the non-standard normalization, for all interaction models, the ion mobility goes to unity, $K \to 1$ at $\epsilon \to 0$.

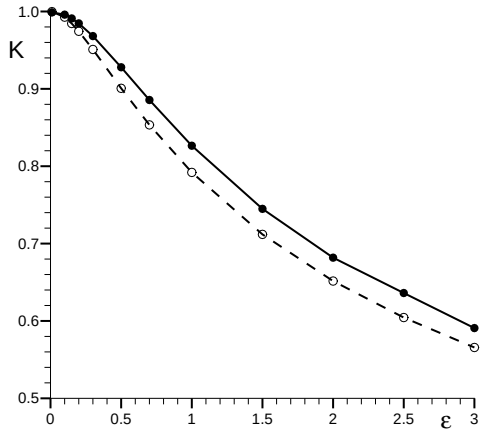

Figure 4: The dependence of the standard mobility K on a strength ϵ for HS–model. For comparison, the dashed curve shows analogius dependences for CEHS–model. The non-standard normalization is used.

If for the models with the constant collision frequency the mobility does not depend on ϵ and equal always unity in the non-standard normalization, for the models with $\lambda = const$, the mobility decreases with an increase in ϵ, such decrease occurring more rapidly than for the hard-sphere model, as one can see in Fig. 4, in the case of CEHS-model.

In a course of investigations, we found a series of the interesting details concerning the behaviour of a heat flow and transverse component of thermal energy, and it was shown that their behaviour changes essentially when transferring from one model of interaction to the other one.

Thus,

1. The comprehensive investigations of the moment method convergence are finished for CEM–model.

2. The way to ameliorate the moment method convergence are outlined involving the transfer to the modified moment method in the cases when the DF does not satisfy the Grad criterion.

3. For several interaction models, the DF calculations were carried out as well as its moments in a domain of the moderate electric fields as well as at the beginning of a region of the strong ones.

6 Sixth quarter

During the 6th Quarter [28] our work was carried out concerning two items.

First, the study of the ion DF under the electric field was continued for the various models of interaction.

Second, a system of the moment equations was deduced for the ions moving in the crossed electric and magnetic fields. In this case, the DF is no longer the axially symmetric one, and the equations for the most general case, when the DF is the 3-Dimension one in a velocity space, should be used.

Namely, in the first part of the 6th quarterly report, the ultimate choice of the standard normalization was made about which we wrote above when describing the Quarter 3. For the models with $\lambda = const$, the definition of the dimensionless electric field was made in a form (54).

Under investigation is the DF evolution at various values of ϵ and different interaction models. The domains of the weak, moderate, and strong electric field, where the Chapman-Enskog method [29] is applicable, were considered in consecutive order. In these cases, there are many known results ready for comparison.

At the very small ϵ, in the Chapman-Enskog method, the DF is sought in a form

$$f(\mathbf{c}) = M(c)(1 + \epsilon\varphi(\mathbf{c})). \tag{85}$$

Here, $M(c)$ is the Maxwellian. We build up the DF via moment method involving very large number of the MEs of the collision integral. It gives an opportunity to build up the DF with a high accuracy, particularly, for the very small values of ϵ. Always, so small ϵ can be found that a correction φ does not depend on ϵ. This φ, built up in such a manner, we call the universal Chapman-Enskog correction.

We calculated the temporal dependences of the universal φ for four models. If, for the CEM–model and the pseudo-Maxwellian molecules , this function is proportional to c (the coefficient of proportionality depends on a time), a tendency to saturation is inherent for CEHS– and HS–models at longer times and high values of c_z. The value of this saturation is about 1.5 times lesser for CEHS–model as compared with HS–model.

When studying the processes under the very weak field, it was shown that the DF building-up requires the DF expansion with the large number of the Sonine polynomials ($N_0 = 128$ and more) although the ion mobility is easily determined with the small amount of the moments (2 or 3).

There is the large section in the 6th quarterly report devoted to another method in which there are no principal limits of the moment method related with the Grad criterion. Note, that the Grad criterion arises at the stage of an expansion in terms of the Sonine polynomials which are orthogonal with the Maxwellian weight. We suppose to carry out the expansion in terms of the spherical harmonics (in the case under consideration, in terms of the Legendre polynomials). This method of solution of the Boltzmann equation goes back to the works [30], [31].

In the axially symmetric case, the DF is represented as (70). At very small ϵ, when the representation (85) is true, a function φ is expanded in terms of the Legendre polynomials, only φ_1 being non-zero. In [30], [31], the linearized Boltzmann equation was considered for HS-model. It was shown that outgoing term of the collision integral corresponds to multiplication of the DF by a function $k(c)$ at

a scattering on equilibrium background gas. Hilbert found a function $k(c)$, in the standard normalization it is as follows

$$k(c) = \frac{e^{-c^2}}{\pi^{1/2}} + (1/2c + c)\Phi(c). \tag{86}$$

Incoming term of the collision integral takes a form of an integral of the DF with a linear kernel, of which explicit expression was obtained in [30] and the expansion in terms of the spherical harmonics in [31]. Using these results for small ϵ, the Boltzmann equation can be easily written as follows

$$\frac{\partial\varphi_1(c,t)}{\partial t} - 2c = \int_0^\infty \tilde{k}_1^+(c,c_1)\varphi_1(c_1,t)c_1^2 dc_1 - k(c)\varphi_1(c,t). \tag{87}$$

For CEM–model, $\tilde{k}_1^+(c,c_1) = 0$, and, $k(c) = 1$at the stationary state, we have

$$\varphi_1(c) = 2c, \tag{88}$$

that coincides with the numerical solution.

If a perturb of the DF from the equilibrium one is not small (in the case under consideration, ϵ is not small), the Boltzmann equation is substituted by the system of equations for $f_l(c)$. For the linear Boltzmann equation (a small admixture on the equilibrium background), we have

$$\frac{D_l f_l(c,t)}{Dt} = \int_0^\infty k_l^+(c,c_1)f_l(c_1,t)c_1^2 dc_1 - k(c)f_l(c,t). \tag{89}$$

Here, D_l/Dt corresponds to the left side of the Boltzmann equation, and the kernels in (87) and (89) are interrelated via a simple relationship

$$\tilde{k}_l^+(c,c_1) = \frac{1}{M(c)}k_c^+(c,c_1)M(c_1). \tag{90}$$

A great advantage of the transfer from the system of the moment equations to the system (89), besides the removal of limitations related with the grad criterion, is the transfer in the outgoing term to a very simple multiplication operator by a function $k(c)$. In the moment method, the outgoing term contribution prevails.

In the case of HS–model, the kernels $k_l^+(c,c_1)$ and $\tilde{k}_l^+(c,c_1)$ are easily expressed via the symmetrical kernels obtained by Hecke [31]. However, for the arbitrary cross sections of interactions, there are no analytical expression for the kernels.

In [1], it was shown that the kernels can be build up via the known MEs. The recurrence relationships with which aid we build up the ME [3] are true for the ME of the incoming term of the collision integral. In [1], it was shown that the kernels can be built up for arbitrary cross sections, even for the non- linear Boltzmann equation.

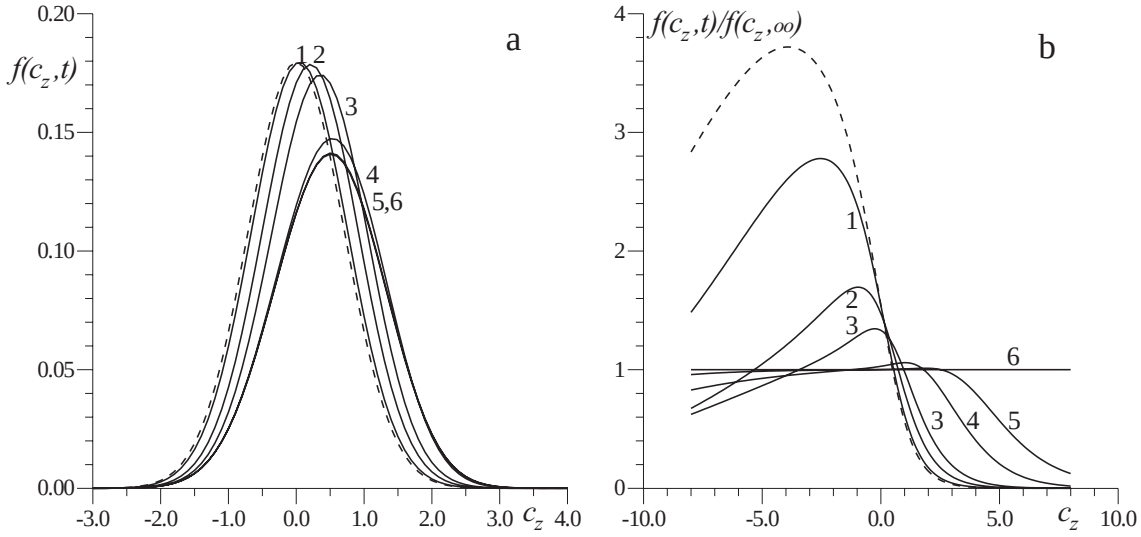

Figure 5: The DF at different times for the hard-sphere model at $\epsilon = 0.5$ in the non-standard normalization. $L_0 = 64, N_0 = 128$. Figure a – $f(c_z, t)$, Fig. b – $f(c_z, t)/f(c_z, \infty)$. The curves: 1 – $t = 0.1$, 2 – $t = 0.5$, 3 – $t = 1$, 4 – $t = 3$, 5 – $t = 5$, 6 – $t = 10$. Dashed curve refers to $t = 0$.

In the linear case of our interest, we have

$$\tilde{k}_l^+(c, c_1) = M(c_1) c^l c_1^l \sum_{r,r_1=0}^{\infty} \frac{1}{\sigma_{r_1,l}} S^r_{l+1/2}(c^2), \Lambda^+_{r,r_1,l} S^{r_1}_{l+1/2}(c_1^2), \tag{91}$$

where $\sigma_{r,l}$ is a square of a norm of the Sonine polynomial.

Thus, when developing this way, there are actual possibilities of an insight into domain of the very strong electric field.

Using a kernel for the hard-sphere model, the estimations were carried out and the asymptotic value of $\varphi_1(c)$ was found at $c \to \infty$. In the standard normalization, it turns out to be equal to 6.

Further, in the first part of the 6th quarterly report, using the standard moment method for several models of interaction, the DF was under study in a domain of the moderate and strong electric field.

As the examples, there are two figures. In Fig. 5a¡ is presented the DF evolution for HS–model at $\epsilon = 0.5$, in Fig. 5b, the same evolution is shown involving the relative DF where $f(c_z)$ refers to the stationary DF, and as the stationary, $f(c_z)$ is chosen at $t = 10$.

In Fig. 6, the stationary DFs are shown for a series of ϵ (HS–model).

The main results of this part can be formulated as follows

1. The DF was built up successfully via the standard method in a wide region of ϵ.

2. An idea to solve the non-standard problem at the instant switch-on of the electric field turns out to be the successful one. Even at the strong field, when

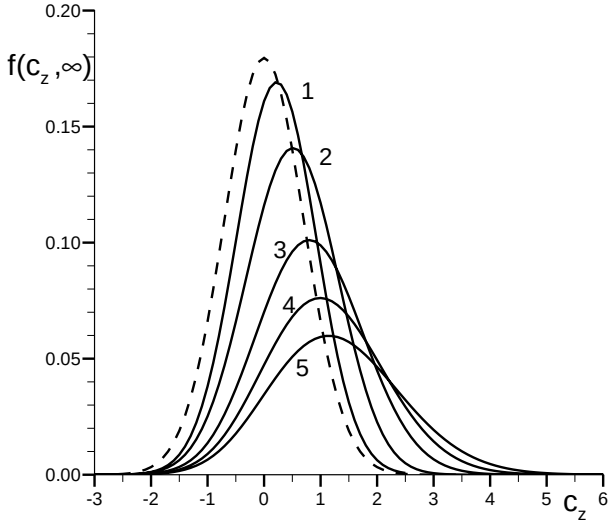

Figure 6: The DF $f(c_z, \infty)$ at different ϵ for the hard-sphere model in the non-standard normalization. $L_0 = 128, N_0 = 128$. The curves: $1 - \epsilon = 0.2$, $2 - \epsilon = 0.5$, $3 - \epsilon = 1$, $4 - \epsilon = 1.5$, $5 - \epsilon = 2$. Dashed curve refers to $t = 0$.

the stationary solution does not satisfy the Grad criterion and can not be built up via the solution of the standard moment system, we succeed to nearly reach the stationary solution.

3. For the first time, it was shown in what manner the DF evolution proceeds in a course of the non-stationary process and how it is affected by the interaction model.

4. The standard moment method is far from being exhausted, and can be used for solving the more complicated problems, namely, concerning the ion kinetics in crossed electric and magnetic fields.

5. The ways to overcome those limits which are relevant for the standard moment method ere outlined. In the second part of the 6th quarterly report, under consideration are the moment equations describing the ion kinetics under crossed electric and magnetic fields when the DF expansion should be performed over a full set of the spherical Hermite polynomials (2).

From the collision integral invariance relative to rotation, the relationships between the MEs of different indices m and i were derived. It was shown that the MEs with arbitrary indices m and i are proportional to the corresponding axially symmetric MEs ($m = 0$, $i = 0$) with the coefficients of proportionality expressed via the Klebsch-Gordan coefficients.

The main result for the case under our consideration can be formulated as follows. **The linear matrix element $K^{rlm}_{r_1 l_1 m_1 000}$ is non zero only at $l_1 = l$, $m_1 = m$,**

and this ME equals the corresponding axially symmetric ME:

$$K^{rlm}_{r_1lm000} = K^{rl}_{r_1l00} = \Lambda_{r,r_1,l}. \tag{92}$$

Thus, the number of the axially symmetric MEs calculated earlier is well suit for solving the problem concerning the ion motion under crossed electric and magnetic fields. Only the number of the equations increases, i.e., at each l, an index m changes from 0 to l.

Under crossed electric and magnetic fields the Boltzmann equation in the dimension form is as follows

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{e}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{H})\cdot\frac{\partial f}{\partial \mathbf{v}} = J_{col}. \tag{93}$$

Under consideration are the spatially uniform case and the mutually perpendicular electric and magnetic fields; for concreteness, the electric field is directed along z-axis and the magnetic one along y-axis. Using the standard notion of the unit vectors along these axes: $\mathbf{k}$ and $\mathbf{j}$, we have in the dimensionless form

$$\frac{\partial f}{\partial t} + (\epsilon\mathbf{k} + h(\mathbf{c}\times\mathbf{j}))\cdot\frac{\partial f}{\partial \mathbf{c}} = J_{col}. \tag{94}$$

Here, the dimensionless magnetic field h is proportional to H

$$h = \frac{e}{m}H\tau, \quad h = \omega_L\tau, \quad \omega_L = \frac{e}{m}H, \tag{95}$$

where ω_L is a Larmor frequency. Remind that an expansion of the angular dependence of the DF is carried out in terms of the real spherical harmonics Y^i_{lm}:

$$Y^0_{lm}(\theta,\varphi) = P^m_l(x)\cos m\varphi, \quad Y^1_{lm}(\theta,\varphi) = P^m_l(x)\sin m\varphi, \quad x = \cos\theta\,, \tag{96}$$

and $P^m_l(x)$ are the associated Legendre polynomials. Here, an angle θ is counted from z-axis, and an angle φ - from a (x,z)-plane.

In the 2nd quarterly report [19], the left sides of the moment equations were built up when supposing the dependence of the external field on a velocity. To obtain an expression for the left side of the moment equations in the case under consideration, the rather cumbersome calculation needs to be carried out.

As a result, the system of the moment equations was built up describing a behaviour of the charged particles under crossed electric and magnetic fields. It takes a form

$$\frac{dC^i_{rlm}}{dt} + \frac{h}{2}((l-m)(l+m+1)C^i_{rlm+1} - d_{m-1}C^i_{rlm-1}) +$$
$$\epsilon\left(\frac{2}{2l+3}r(l+m+1)C^i_{r-1l+1m} - \frac{2}{2l-1}(l-m)C^i_{rl-1m}\right) = \sum_{r_1}\Lambda_{r,r_1,l}C^i_{r_1lm}, \tag{97}$$
$$d_m = 1 + \delta_{m0}.$$

7 Seventh quarter

The main problem to be solved in the 7th quarter is to build up the analytical solutions for several first moments in crossed electric and magnetic fields [32].

Under investigation is a transient process after the switch-on of the electric field under existing magnetic field being perpendicular to the electric one.

At the symmetric initial conditions under our consideration, an index i in a system (97) can be only zero and, usually, it is omitted.

Beside of the analytical solutions, a code for numerical solution was developed (97). The analytical solutions for the moments C_{rlm} were successfully built up for the molecules of the Maxwellian type, i.e., those of which cross section is inversely proportional to the relative velocity and its angular dependence is the arbitrary one. The non-standard normalization is used, all the MEs being divided by the normalization coefficient which is chosen to satisfy the condition when the mobility equals unity and coincides with that for CEM-model at zero magnetic field and very weak electric field. For the Maxwellian molecules with arbitrary angular part of the scattering cross section, the matrix corresponding to the linear collision integral is diagonal, all diagonal MEs are negative and do not depend on an index m. For such molecules, the renormalization coefficient turns out to be equal to an initial Λ_{001}, i.e., in the new normalization, $\Lambda^N_{001} = -1$. We introduce a new notion

$$\lambda_{rl} = -\Lambda^N_{rrl}. \tag{98}$$

For the arbitrary Maxwellian molecules in the non-standard normalization, $\lambda_{01} = 1$.

Note, that, with a change in the normalization, the time and electric field units change also change as well as that of the magnetic field in the case under consideration.

For concreteness, the pseudo-Maxwellian molecules were considered with the isotropic angular part of the scattering cross section.

If, with no magnetic field, a drift velocity

$$\mathbf{u}_d = \int f \mathbf{c} d^3 c$$

has a single component u_{dz} related with the moment C_{010} via a formula $u_{dz} \equiv u_\parallel = C_{010}/2$, at the mutually perpendicular electric and magnetic fields, another component of a drift velocity vector u_{dx} arises directed along x–axis being perpendicular to the directions of the electric and magnetic fields and equals to $u_{dx} \equiv u_\perp = C_{011}/2$. Namely, a component of the drift velocity u_{dx} determines a Hall current. Besides the drift velocity vector, a mobility vector $\mathbf{K}$: $u_{dz} = K\epsilon$, $u_{dx} = K_{hal}\epsilon$ can be introduced.

Usually, to characterize a direction of a drift velocity, a so called Lorentz angle φ is introduced which is determined as follows

$$\tan\varphi \equiv \frac{u_\perp}{u_\parallel}. \tag{99}$$

The value of the drift velocity is determined as

$$u_d = (u_\parallel^2 + u_\perp^2)^{\frac{1}{2}} \tag{100}$$

From (97), we outline the equations for C_{010} and C_{011}

$$\frac{dC_{010}}{dt} + hC_{011} - 2\epsilon C_{000} = \Lambda^N_{001} C_{010} + \sum_{r1\neq 0} \Lambda^N_{0,r_1,1} C_{r_1 10},$$

$$\frac{dC_{011}}{dt} - hC_{010} = \Lambda^N_{001} C_{011} + \sum_{r1\neq 0} \Lambda^N_{0,r_1,1} C_{r_1 11}. \tag{101}$$

Here, C_{000} does not depend on time and set to be equal unity. In a system (101) under sign of summation, the moments C_{r1m} enter at $m = 0, 1$ and $r \neq 0$, and, in general case, to calculate the temporal dependence of a drift velocity, a full system of the moment equations is needed to be solved.

In the case of the diagonal interaction matrix , the equations for C_{010} and C_{011} are split out and, taking in mind (98) from (101), we have

$$\frac{dC_{010}}{dt} + hC_{011} - 2\epsilon = -C_{010},$$

$$\frac{dC_{011}}{dt} - hC_{010} = -C_{011}. \tag{102}$$

Let till $t = 0$ there was a magnetic field h and no electric field, and at $t = 0$ the electric field ϵ experiences a sharp switch-on.

As the magnetic field itself does not affect the equilibrium DF, the DF continues to be Maxwellian till a switch-on of the electric field¿ and $C_{0lm} = 0$ at $l \neq 0$. Hence, a system (102) has zero initial conditions. Apply the Laplace perform $g^p = \hat{L}(g(t)) = \int\limits_0^\infty e^{-pt} g(t) dt$ to (102)

$$(p+1)C^p_{010} + hC^p_{011} = \frac{2\epsilon}{p}$$

$$-hC^p_{010} + (p+1)C^p_{011} = 0. \tag{103}$$

This system has a determinant $D = (p+1)^2 + h^2$ and its solution takes a form

$$C^p_{010} = \frac{2(p+1)\epsilon}{pD}, \quad C^p_{011} = \frac{2h\epsilon}{pD}. \tag{104}$$

Carrying out the inverse Laplace perform, we obtain

$$C_{010}(t) = \frac{2\epsilon}{1+h^2}\Big(1 + e^{-t}(h\sin ht - \cos ht)\Big),$$

$$C_{011}(t) = \frac{2h\epsilon}{1+h^2}\Big(1 - e^{-t}\Big(\frac{\sin ht}{h} + \cos ht\Big)\Big). \tag{105}$$

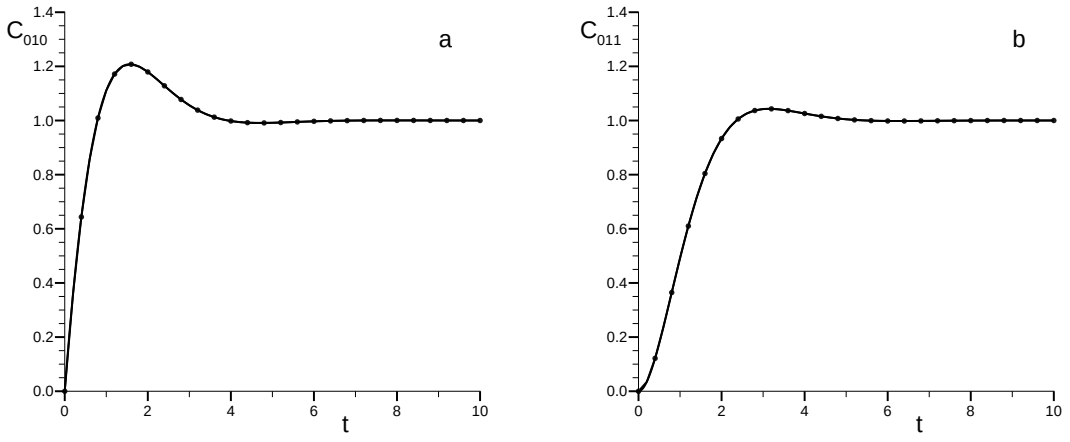

Figure 7: The temporal dependences of the moments C_{010} and C_{011} for arbitrary Maxwellian molecules, $\epsilon = 1$, $h = 1$ in the non-standard normalization. Solid curves present the analytical results, points are the numerical solutions of the system of the moment equation. Figure $a - C_{110}$, $b - C_{111}$

The comparison of these analytical results with those obtained via the numerical solution of the system of the moment equations at $\epsilon = 1$ and $h = 1$ is presented in Fig. 7. One can see that the temporal dependences have an oscillating component. When h increases, the oscillation feature of the transient process evolves more evidently.

At the stationary state, from (105) at $t \to \infty$ we obtain

$$C_{010} = \frac{2\epsilon}{1+h^2}, \quad C_{011} = \frac{2h\epsilon}{1+h^2}. \tag{106}$$

In papers [33], [8], an expression for a drift velocity at the stationary state is considered for the constant collision frequency, i.e., at the diagonal interaction matrix.

If we transfer to the our notations, the results [33] are totally coincide with (106). At the stationary state, the Lorentz angle (99) and the drift velocity (100)are determined as follows

$$\tan\varphi = h,$$
$$u_d = \frac{\epsilon}{(1+h^2)^{\frac{1}{2}}} = \epsilon\cos\varphi = \frac{\epsilon}{h}\sin\varphi. \tag{107}$$

If we go to the dimensional variables, it is seen that the oscillation frequency of the drift velocity coincides with the Larmor one in the non-stationary process.

In the 7th quarterly report, the expression of the principal physical moments is obtained via the moments C_{rlm} and it is shown that for their calculation C_{rlm} with $l \leq 2$ and $r \leq 1$ with are needed. These formulas were obtained in the quarterly report and do not presented here. Besides a drift velocity, there are considered a

pressure tensor and a vector of a heat flow as the physical moments; a pressure tensor has a component p_{xz} besides of the diagonal components p_{xx}, p_{yy} and p_{zz}. In the case of the diagonal interaction matrix, further splitting out occurs of the system of the moment equations for C_{rlm}. At fixed r and l, there are $l+1$ equations for $0 \le m \le l$. These equations are solved, also, using the Laplace transform and convolution theorem.

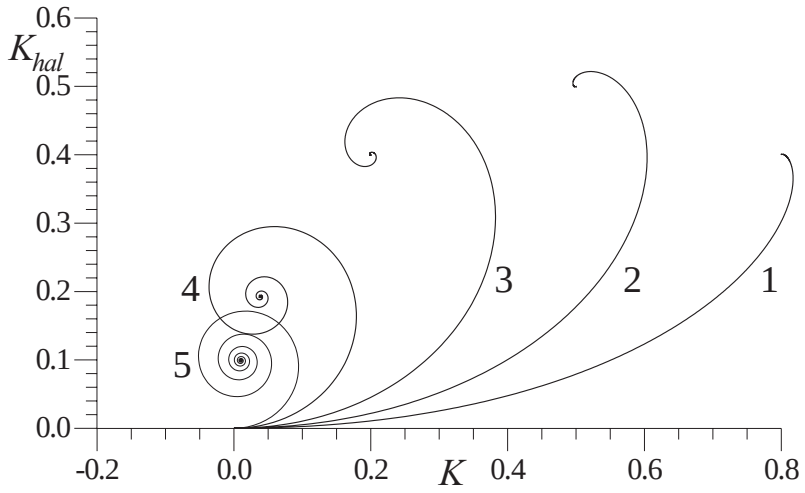

Figure 8: The trajectories of a drift velocity for several values of h. Curve 1 refers to h=.5, 2 – $h = 1$, 3 – $h = 2$, 4 – $h = 5$, 5 – $h = 10$.

As a result, the temporal dependences of all physical moments are built up. The mobility vector behaviour is of interest. As one can see in Fig. 8, it oscillates and, on a (K, K_{hal})-plane, it represents a twisting spiral, the oscillation amplitude and the number of the spiral turns increase.

Thus, for the first time, Report [32] presents the investigations of the non- stationary transient processes evolving into an ion admixture on the background of the atoms at a sharp switch-on of the electric field under the magnetic field (generally, for the pseudo-Maxwellian molecules).

A code was created to solve a system of the integro-differential equations involving the ion interaction with the electric field as well as the magnetic one. It was shown that the system of the moment equations is divided on the subsystems for C_{rlm} with fixed r and l and $0 \le m \le l$.

Involving the Laplace transform, this subsystem of the differential equations is converted into a subsystem of algebraic equations. It was proved that the determinant of this subsystem has the roots $\lambda_{rl} \pm ikh$, where k are integers of which parity coincides with that of l.

It was maintained that, in the case of the diagonal interaction matrix (Maxwellian molecules with arbitrary angular part of the scattering cross section), the subsystems of the equation aligned in a recurrently solvable sequence. It signifies that, if one moves along this sequence, solving the subsystem for C^p_{rlm}, all $C^p_{r'l'm'}$ with $r'l'$ combinations distinct from $r'l'$ were already known.

The analytical solutions for the moments C_{rlm} with the small values of r, l. It was shown in what manner were expressed the principal physical moments via these moments. It was found that, the oscillations arose during the transient stage of the process evolution. The principal harmonic at a Larmor frequency as well as higher harmonics at the frequencies, which are multiples of the principal one, were excited.

Of exceptional interest is excitation of oscillations of the Hall current and a current along the electric field which can be observed in a particular experiment and use for diagnostics.

Via the analytical formulas as well as the numerical solution of the system of the moment equations with aid of our code, the large amount of the results were obtained to describe evolution of both the DF moments evolution and physical ones. The fine coincidence of the analytical and numerical results was obtained.

A detailed analysis of a behaviour of physical moments is presented at the transient stage under different values of the magnetic field h and dependences of the stationary values of these moments on h.

8 Eighth quarter

The 8th quarterly report [34] describes the calculations of the ion physical moments at the mutually perpendicular electric and magnetic fields for various interaction models. As before, under investigations is a transient process after the instant switch-on of the electric field. The 7th quarterly report [32] develops the code for solving the system of the moment equations in the case arbitrary interaction model. It is shown that the numerical and analytical results coincide in practice, for the Maxwellian molecules, this proving the involved code.

The 8th quarterly report uses widely the developed code to study the transient processes at the various ion-atom interaction models. When calculating, the non-standard normalization was used.

First of all, the calculations for the interaction models with the constant collision frequency (in this case, the interaction matrix is diagonal) and a comparison of the processes for the pseudo-Maxwellian molecules and CEM-model is undertaken. It is shown that, in the case of the diagonal interaction matrix, the physical moments are proportional to the different degrees of the dimensionless electric field ϵ.

$$K = \frac{u_{dz}}{\epsilon}, \quad K_{hal} = \frac{u_{dx}}{\epsilon}, \quad \bar{p}_{xx} = \frac{p_{xx} - .5}{\epsilon^2}, \quad \bar{p}_{yy} = \frac{p_{yy} - .5}{\epsilon^2}, \quad \bar{p}_{zz} = \frac{p_{zz} - .5}{\epsilon^2},$$

$$\bar{p}_{xz} = \frac{p_{xz}}{\epsilon^2}, \quad \bar{\mathbf{q}} = \frac{\mathbf{q}}{\epsilon^3}, \tag{108}$$

which do not depend, already, on ϵ.

For all models with the non-diagonal interaction model in a domain of the weak electric field, the physical moments turn out to be proportional the different degrees of ϵ. However, if a drift velocity (mobility) and pressure tensor depend on ϵ in the

manner as in the case of the diagonal matrix, a heat flow turns out to be proportional to ϵ but not ϵ^3.

Besides the models with $\tau = const$, the systematic calculations was carried out for the following interaction models: 1. HS-model, the hard-sphere model. 2. CEHS-model, the charge exchange with the scattering cross section independent on the velocity. 3. Coulomb interaction. For all models, the reduced physical moments are determined as follows

$$K = \frac{u_{dz}}{\epsilon}, \quad K_{hal} = \frac{u_{dx}}{\epsilon}, \quad \bar{p}_{xx} = \frac{p_{xx} - .5}{\epsilon^2}, \quad \bar{p}_{yy} = \frac{p_{yy} - .5}{\epsilon^2}, \quad \bar{p}_{zz} = \frac{p_{zz} - .5}{\epsilon^2},$$

$$\bar{p}_{xz} = \frac{p_{xz}}{\epsilon^2}, \quad \bar{\mathbf{q}} = \frac{\mathbf{q}}{\epsilon}. \tag{109}$$

We call the weak electric field that of $\epsilon \leq 0.1$. In a subregion of the weak field $\epsilon < 10^{-2}$, a change in ϵ, for all models under consideration does not results in any changes in the temporal dependences of the reduced physical moments. The reduced physical moments at the very weak electric fiels ($\epsilon < 10^{-2}$) are called by us **the standard reduced physical moments**.

For all model except $\tau = const$, with an increase in the electric field, the reduced physical moments experience the changes. Nevertheless, the reduced physical moments turn out to be advisable to study when being built up not only at the small ϵ. The 8th quarterly report gives the results for three interaction models, 5 values of ϵ variants, 15 overall, being considered in the domains of the weak and moderate electric field. In every variant, the temporal dependences of 9 physical moments are presented, being built up at 6 values of h ($h = 0.1$, $h = 0.5$, $h = 1$, $h = 2$, $h = 5$, $h = 10$). The total amount of dependences is above 800. All this set of the curves can be called **ATLAS** of the temporal dependences of the physical moments for the ions at the crossed electric and magnetic fields under a sharp switch-on of the electric field.

This atlas is the principal part of the full report for the Quarter 8.

It was found that, at the weak magnetic field in the domains of the moderate and strong electric field, a distinction from the reduced physical moments is pronounced, being especially large for the heat flow. However, with an increase in the magnetic field, this distinction diminishes and, at $h = 5$, 10, the reduced physical moments coincide with the standard ones. At the weak magnetic field ($h = 0.1$, $h = 0.5$), for the models $\lambda = const$ both the mobility and the tensor components turn out to be smaller that the standard ones. A completely different situation occurs in the case of the reduced physical moments in the case of the Coulomb interaction. In Fig. 9, the mobility behaviour is shown.

For this model, the mobilities turn out to be higher than the standard ones. Moreover, at a small h and $\epsilon = 0.5$, there is a tendency to an unlimited increase; at the smaller values of ϵ, this effect does not almost occur, but, at $\epsilon > 0.5$, it is even more pronounced. Along an analogy with the electron running-off, we call it as ion running-off.

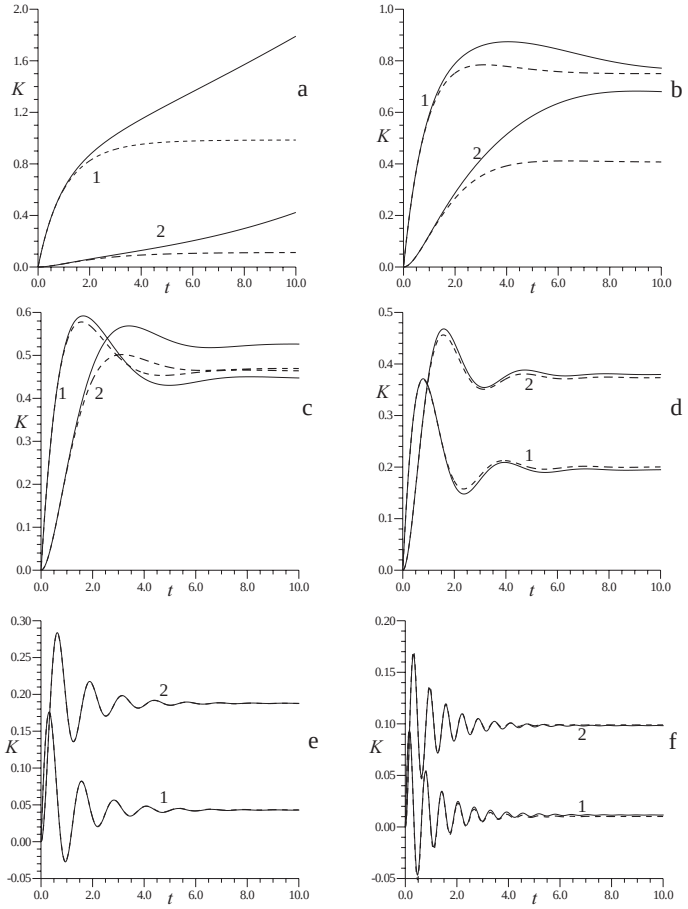

Figure 9: Dependences $K(t)$ (Curve 1) and $K_{hal}(t)$ (Curve 2) for the Coulomb interaction at $\epsilon = 0.5$ and different values of h. The dashed curves refer to the standard mobility. Figure 1 a – $h = .1$, b –$h = .5$, c –$h = 1$, d – $h = 2$, e –$h = 5$, f – $h = 10$.

This effect is a consequence of a very strong decrease in the scattering cross section with an increase in a relative velocity($\sigma \sim g^{-4}$).

From Fig. 9, one can see, also, that the running-off effect is suppressed by the magnetic field. At the strong magnetic field ($h = 5$ or $h = 10$) the reduced mobility coincides with the standard one, the moment method convergence being restored. A good convergence at the strong magnetic field is observed for all interaction models, it makes possible, at large h, an advance into the strong electric field domain, i.e., in the very region being non-calculable with standard moment method at no magnetic field.

Thus, in Quarter 8, the calculation of the evolution process concerning the physical moments for various laws of interaction from the most hard interaction potential for the hard-sphere model to the softest, i.e., Coulomb interaction.

Below, the conclusions made on a base of this study:

1. The notion of the reduced physical moments (see, formulas (108) and (109))

is introduced. The reduced moments in a domain of the weak field do not depend on ϵ and are used as the standard reduced moments.

2. It is found that the behaviour of the components of a pressure tensor is determined, generally, by the angular part of the scattering cross section and that of the components of a heat flow by the velocity dependence of the scattering cross section.

3. It is shown that, in a domain of the weak electric field, the mobilities in a direction of the electric field K as well as that perpendicular to the electric and magnetic field, K_{hal}, are similar in their behaviour at all the scattering cross sections.

4. In a domain of the moderate and strong electric fields, for a series of the interaction cross section, the ATLAS of the non-stationary characteristics of the physical moments is compiled, these characteristics are presented in a wide range of ϵ and h changes. 5. It is shown that, in the case of the Coulomb interaction at a small h, the ion running-off takes place and this process becomes more pronounced with an increase in ϵ.

The running-off process at the Coulomb interaction is suppressed by the strong magnetic field.

6. It is shown that, for any interaction models at the strong magnetic field, all the physical moments coincide with the standard ones.

7. The moment method can be used successfully for calculation of the non-stationary ion process at the crossed electric and magnetic fields in a domain of the very strong electric fields if the magnetic field is sufficiently strong(ϵ and h are commensurable).

9 Ninth quarter

In the 9th quarterly report [35], the calculation of the ion DF evolution at the mutually perpendicular electric and magnetic fields were performed for different interaction models. The calculations were based on a code concerned with evolution of the moments C_{rlm} which was developed in Quarter 7 [32]. The problems solved in Quarters 9 and 10 involve the DF restoring via its moments as well as descriptive DF representation.

With no magnetic field, the electric field switch-on directed along z results in stretching the DF along c_z. The DF, at any time, has the axial sysmmetry relative to c_z-axis, i.e., depends on two variables c_z- and $c_\rho = \sqrt{c_x^2 + c_y^2}$.

There are more complication when dealing with the DF shape as well as the descriptive representation of its evolution at the switch-on of the mutually perpendicular electric and magnetic field. A direction of the magnetic field is matched with y-axis, Due to ion rotation about y axis, there is a strong rebuilding of the DF in (c_x, c_z)-plane, the symmetry relative c_y substitution with $-c_y$ being conserved. A departure from the maxwellian dependence relative to $|c_y|$ turns out to be neglecting. So, we follow the DF evolution when building up $f(c_z, c_x)$ at $c_y = 0$.

At such image, in a fixed time, the DF is represented by certain surface in 3D space. The DF expansion can be performed in two steps: first, an expansion in terms of the spherical harmonics:

$$f(\mathbf{c}) = \sum_{l,m} Y_{lm}(\Theta, \varphi) f_{lm}(c), \tag{110}$$

second, in terms of the Sonine polynomials

$$f_{lm}(c) = M(T_0, c^2) \sum_{r=0}^{\infty} C_{r,l,m} c^l S^{(r)}_{l+1/2}(c^2) \tag{111}$$

In the problems under consideration, the expansion is performed in terms of the cosine spherical harmonics, so, in (110), an index i is omitted. Knowing C_{rlm}, using (111), one can build up all $f_{lm}(c)$, then, using (110), restore the DF $f(\mathbf{c})$.

If $\epsilon \leq 1$, the DF can be expanded in a series in terms of the powers of ϵ. As a preliminary, we extract the Maxwellian distribution from the DF, passing to $\tilde{f}(\mathbf{c})$ via a formula

$$f(\mathbf{c}) = M(T_0, c^2) + \tilde{f}(\mathbf{c}) \tag{112}$$

Clearly, for the expansion coefficients of the spherical harmonics we have

$$\tilde{f}_{00}(c) = f_{00}(c) - M(T_0, c^2), \quad if\ l = 0, \quad \tilde{f}_{lm}(c) = f_{lm}(c) \quad if\ l \neq 0 \tag{113}$$

In the 9th quarterly report, the dependences $\tilde{f}_{lm}$ on ϵ were studied numerically in a region $\epsilon \ll 1$ at the mutually perpendicular electric and magnetic fields, It was found that, independently on the interaction model, $f_{lm} \sim \epsilon^l$ at $l \neq 0$ and $\tilde{f}_{00} \sim \epsilon^2$. The functions Φ_{lm} were determined as follows

$$\Phi_{lm} = \frac{f_{lm}}{\epsilon^l} \qquad l > 0,$$

$$\tilde{\Phi}_{00} = \frac{\tilde{f}_{00}}{\epsilon^2} \tag{114}$$

In a domain of the very weak electric field, these functions do not depend on ϵ.

The calculations of Φ_{lm} functions were carried out for different interaction models. Graphically, they do not differ at $\epsilon \leq 10^{-2}$. Under an increase in ϵ, these function change. The report presents the evolution of several such functions at different values of the electric and magnetic field.

In the case of the weak electric field, when expanding the DF in terms of ϵ powers, there can be a restriction by the first term

$$f(\mathbf{c}) = M(T_0, c^2)(1 + \epsilon\Phi(\mathbf{c})) \tag{115}$$

A function $\Phi(\mathbf{c})$ independent on ϵ in a domain of the weak electric field is called the universal DF, being under study in the report. Fot it, we have

$$\Phi(\mathbf{c}) = \sum_{m=0}^{1} \Phi_{1m}(c) Y_{1m}(\Theta, \varphi), \tag{116}$$

A polar angle φ in (c_x, c_y)-plane is counted from the direction of c_x-axis. Hence, in $c_y = 0$-plane, an angle $\varphi \equiv 0$ and

$$\Phi(c_x, c_z) = \Phi_{10}(c)\cos(\Theta) + \Phi_{11}(c)\sin(\Theta). \tag{117}$$

Denote

$$R(c) = \sqrt{(\Phi_{10}(c))^2 + (\Phi_{11}(c))^2}. \tag{118}$$

and determine an angle $\alpha(c)$ as follows

$$\alpha(c) = \arctan\frac{\Phi_{11}(c)}{\Phi_{10}(c)} \tag{119}$$

Then, $\Phi_{11}(c) = R(c)\sin\alpha(c)$ and

$$\Phi(c_x, c_z) = R(c)\cdot\cos(\Theta - \alpha(c)). \tag{120}$$

From (120) one xan see that the DF $\Phi(c_x, c_z)$ is uniquely determined by two functions $R(c)$ and $\alpha(c)$ which are called its characteristics. At fixed c, the maximum of $\Phi(\Theta)$ is observed at $\Theta = \alpha$, being equal $R(c)$, and the minimum being at $\Theta = \alpha + \pi$. Additionally, we have

$$\Phi(-c_x, -c_z) = -\Phi(c_x, c_z) \tag{121}$$

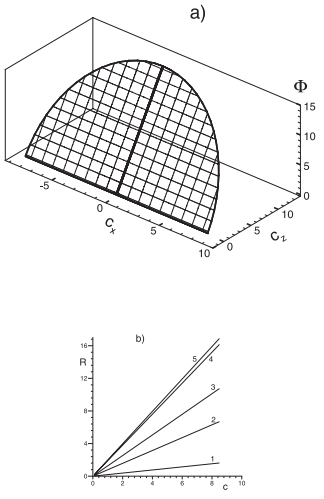

Figure 10: The DF $\Phi(c_x, c_z)$ at $\epsilon < 10^{-2}$ and its characteristics, CEM-model, $h = 0$. Figure a is 3D- representation of the stationary DF ($t \to \infty$). Figure b represents the characteristics $R(c)$ at different times, Curve 1 refers to $t = 0.1$, 2 – $t = 0.5$, 3 – $t = 1$, 4 – $t = 3$, 5 – $t = 5$.

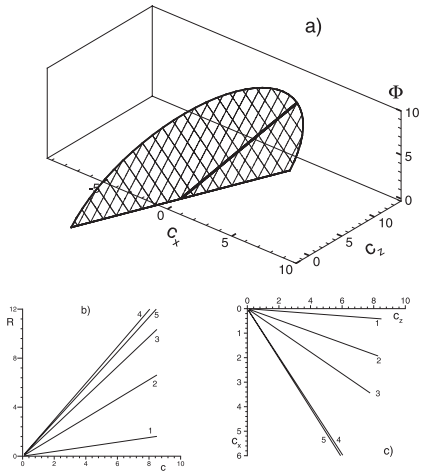

Figure 11: The DF $\Phi(c_x, c_z)$ at $\epsilon < 10^{-2}$ and it's characteristics, CEM– model, $h = 1$. Figure a is 3D– representation of the stationary DF ($t \to \infty$). Figure b and Figure c are projection of α-line at the c_x, c_z-plane. Curve 1 refers to $t = 0.1$, 2 – $t = 0.5$, 3 – $t = 1$, 4 – $t = 3$, 5 – $t = 5$.

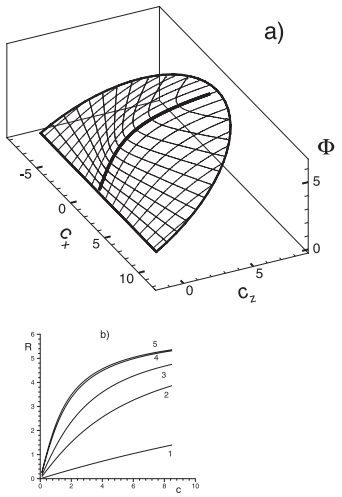

Figure 12: The DF $\Phi(c_x, c_z)$ at $\epsilon < 10^{-2}$ and it's characteristics, HS-model, $h = 0$. Figure a refers to 3D-representation of the stationary DF ($t \to \infty$). Figure b presents the characteristics $R(c)$ at different times. Curve 1 refers to $t = 0.1$, 2 – $t = 0.5$, 3 – $t = 1$, 4 – $t = 3$, 5 – $t = 5$.

This property gives a possibility to be restricted by the DF building-up only at

a half-plane, e.g., $c_z \geq 0$.

A dependence $\alpha(c)$ is a projection of a line of maxima of the DF in the polar coordinates on (c_x, c_z)-plane.

It should be noted that, with no magnetic field, $\alpha(c) \equiv 0$.

When graphically representing the DF, 3D picture is built up in which there is $\Phi(c_x, c_z)$ and a thick line represents the line of maxima. 3D figure is built up for the stationary state corresponding to $t \to \infty$. For obtaining a representation of evolution process, additionally, for several times, the characteristics of a function $\Phi(c_x, c_z)$ are shown, i.e., the curves $R(c)$ and projections α-line at a plane (c_x, c_z); these projections are not built up at no magnetic field ($h = 0$).

Several examples of the calculations are shown in Figs. 10 – 13

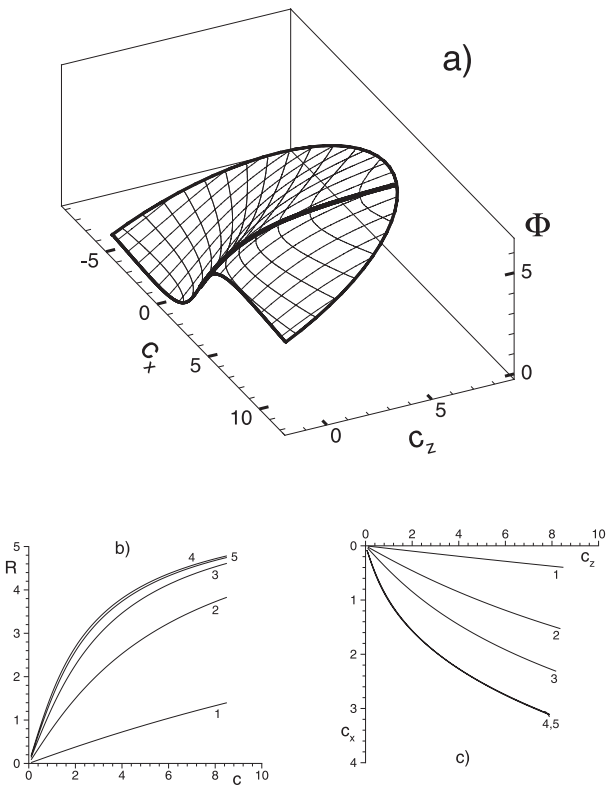

Figure 13: The DF $\Phi(c_x, c_z)$ at $\epsilon < 10^{-2}$ and its characteristics, HS-model, $h = 1$. Figure a refers to 3D-representation of the stationary DF ($t \to \infty$). Figures b and c present the DF characteristics at different times. Figure b refers to $R(c)$, Figure c presents the projections of α-line at a palne c_x, c_z. Curve 1 refers to $t = 0.1$, 2 – $t = 0.5$, 3 – $t = 1$, 4 – $t = 3$, 5 – $t = 5$.

10 Tenth quarter

In the 10th quarterly report [36], the calculations of DF evolution were continued for different interaction models at the instant switch-on of the electric field and the mutually perpendicular electric and magnetic fields, Being distinct from Quarter 9 [35], when the electric field was set to be weak ($\epsilon \leq 10^{-2}$), here, it was considered the significantly stronger electric field up to $\epsilon = 1$ and even more. As in the previous report, the expansion in terms of the real spherical harmonics was of wide use.

If, under the weak electric field at fixed velocity c, the DF dependence on an angle of rotation in (c_x, c_z)-plane is of cos-character, now, this dependence is more complicated and changes when changing both c and magnetic field h. It changes strongly when changing the interaction model.

An increase in the electric field with no magnetic field results in stretching the DF along c_z-axis, and a sharp maximum arises on the angular dependence. At the no so strong magnetic field, the sharp maximum is conserved, a change is only a direction of the stretching. The maximum height can experience a change of several orders of value when changing c. In this case, all that results in a loss of clearness of 3D-representation of the DF, which was used in the 9th quarterly report [35] when studying the DF at small ϵ.

So, to represent the DF, the angular dependences in (c_x, c_z)-plane were built up at several values of a velocity module $c = \sqrt{c_x^2 + c_z^2}$. An angle ψ was counted from c_z-axis and changed from $-pi$ to pi.

In the 10th quarterly report, the detailed calculation were carried out, being illustrated by a number of the figures for three interaction models: HS-model, CEM-model, and the Coulomb interaction. Here, as an example, we present only four figures (14– 17), corresponding to two extreme cases: the hardest model (HS–model) and the softest (Coulomb interaction). Only data are presented for one value of the electric field, $\epsilon = 1$, and two values of the magnetic field, $h = 0$ and $h = 1$.

In these figures, one can see that the magnetic field results in a rotation of the location of the maximum, being more pronounced in a domain of the high velocities. Thus, e.g., a transfer from HS–model to the Coulomb interaction results in about two order higher maximum at $c = 5$.

In the case of the Coulomb interaction, a point under consideration was in what manner the running-off influences the DF evolution. This effect is observed at not so strong transverse magnetic field and appears in a constant and rather rapid increase of the maximum of a dependence $f(\psi)$ in a domain of high velocities. This problem is rather complicated and, when using the standard moment method, we come up quite quickly (at not so long time) against a situation of no convergence of the DF.

To ameliorate the convergence, a transfer to other basis was undertaken, i.e., instead of an expansion about the weight Maxwellian with a temperature of atoms it was used the Maxwellian of higher temperature.

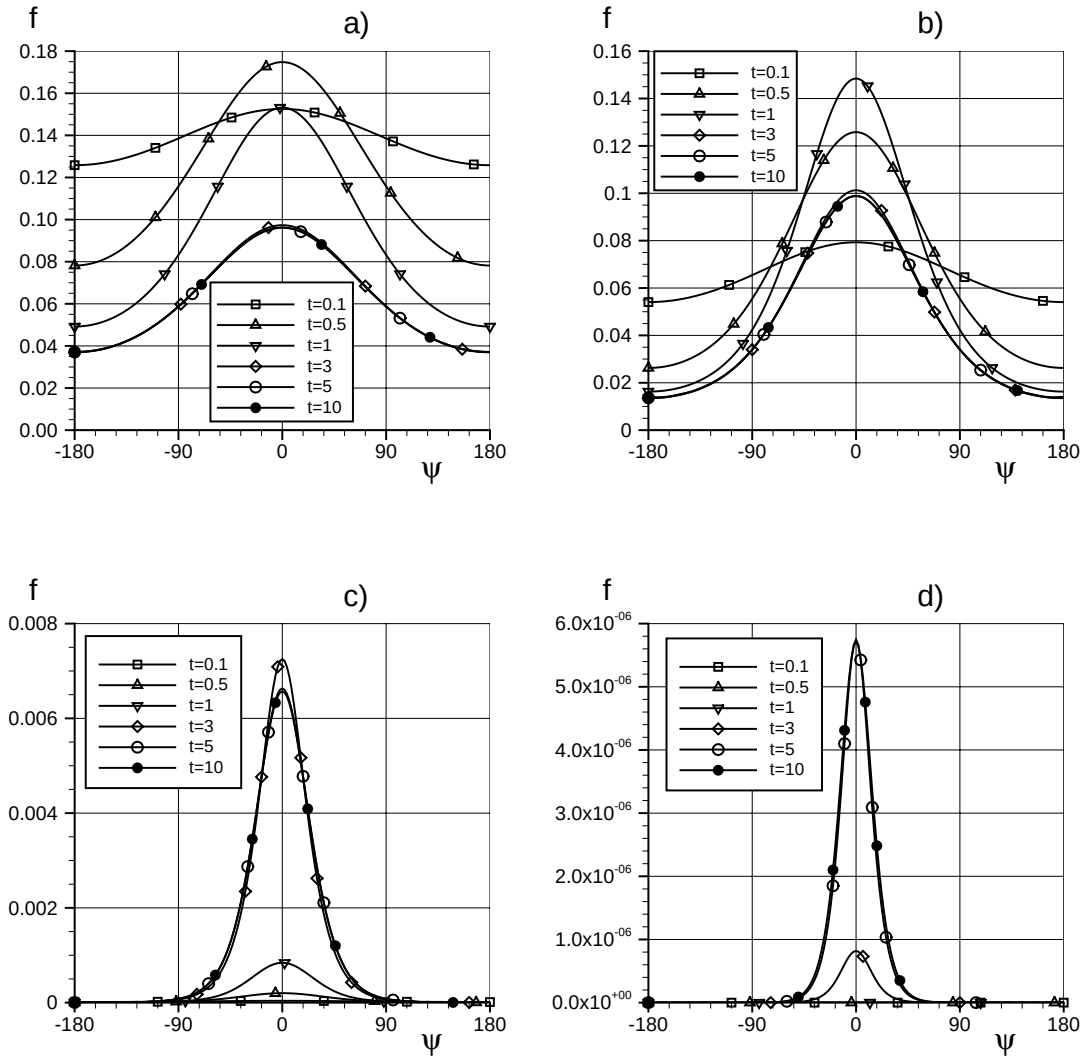

Figure 14: The dependences $f(\psi)$ at different times at different values of the velocity module c for HS-model; $\epsilon = 1$, $h = 0$. Figure a) refers to $c = 0.5$; b) – $c = 1$; c) – $c = 3$; d) – $c = 5$.

Represent the DF in two bases of the temperatures T_0 and T_1:

$$f(v,t) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} C^0_{k,l}(t) M(v,T_0) S^k_{1/2}(v^2/T_0) c^l P_l =$$

$$= \sum_{l=0}^{\infty}\sum_{r=0}^{\infty} C^1_{r,l}(t) M(v,T_1) S^r_{1/2}(v^2/T_1) c^l P_l. \qquad (122)$$

The vectors $\mathbf{C}^0$ and $\mathbf{C}^1$, when transferring from a basis T_0 to a basis T_1, change in

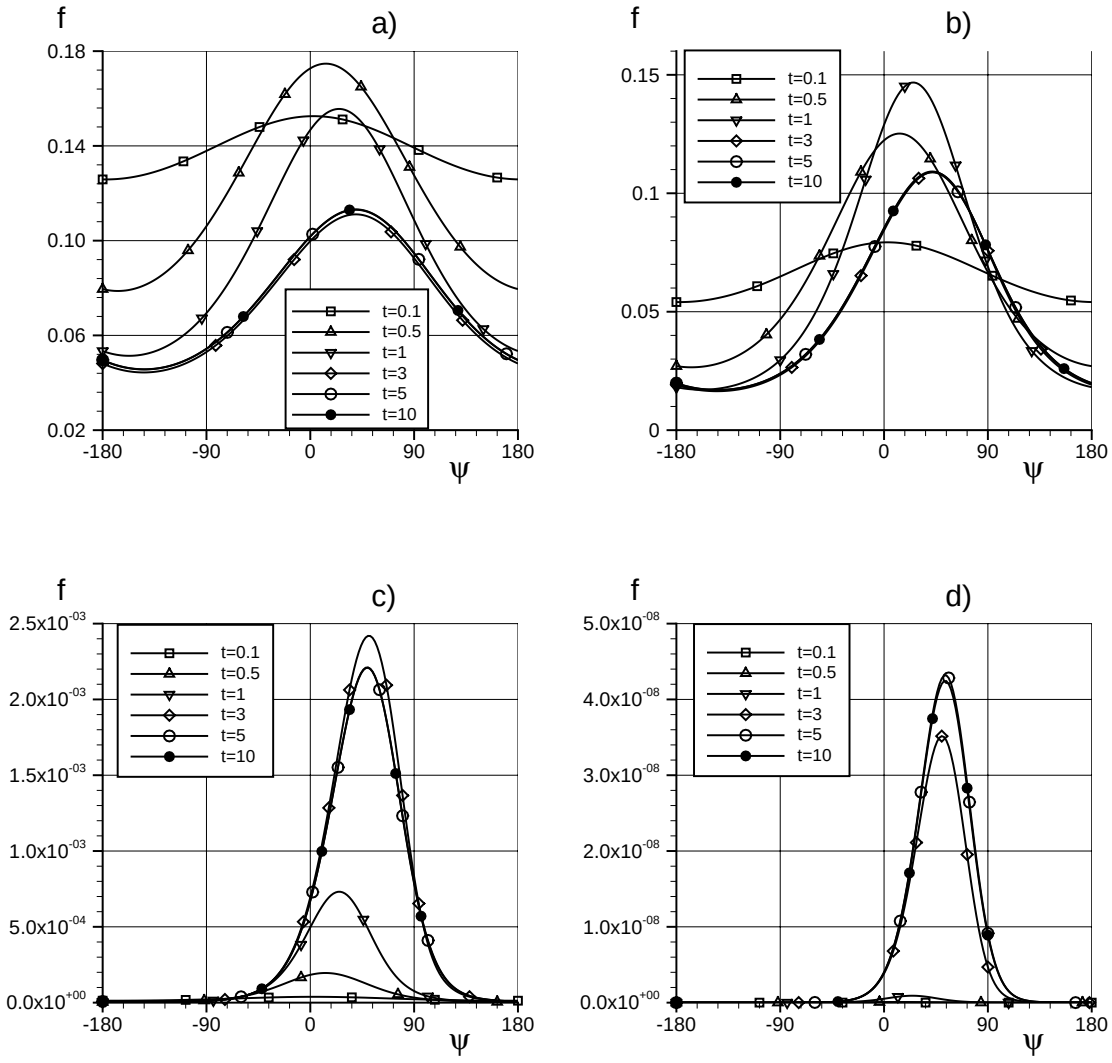

Figure 15: The dependences $f(\psi)$ for different times at different values of the velocity module c for HS-model; $\epsilon = 1$, $h = 1$. Figure a) refers to $c = 0.5$; b) – $c = 1$; c) – $c = 3$; d) – $c = 5$.

a manner as follows

$$C^1_{r,l} = \sum_{k=0}^{\infty} D_{r,k,l}(T_1, T_0) C^0_{k,l}, \tag{123}$$

A transfer matrix $D_{r,k,l}$ is as follows

$$D_{r,k,l} = \frac{(-1)^{r+k} k!}{r!} \frac{T_0^{r+l/2}}{T_1^{k+l/2}} \frac{(T_1 - T_0)^{k-r}}{(k-r)!}. \tag{124}$$

Denote a ratio of the temperature of these bases as $RT = T_1/T_0$. We call the

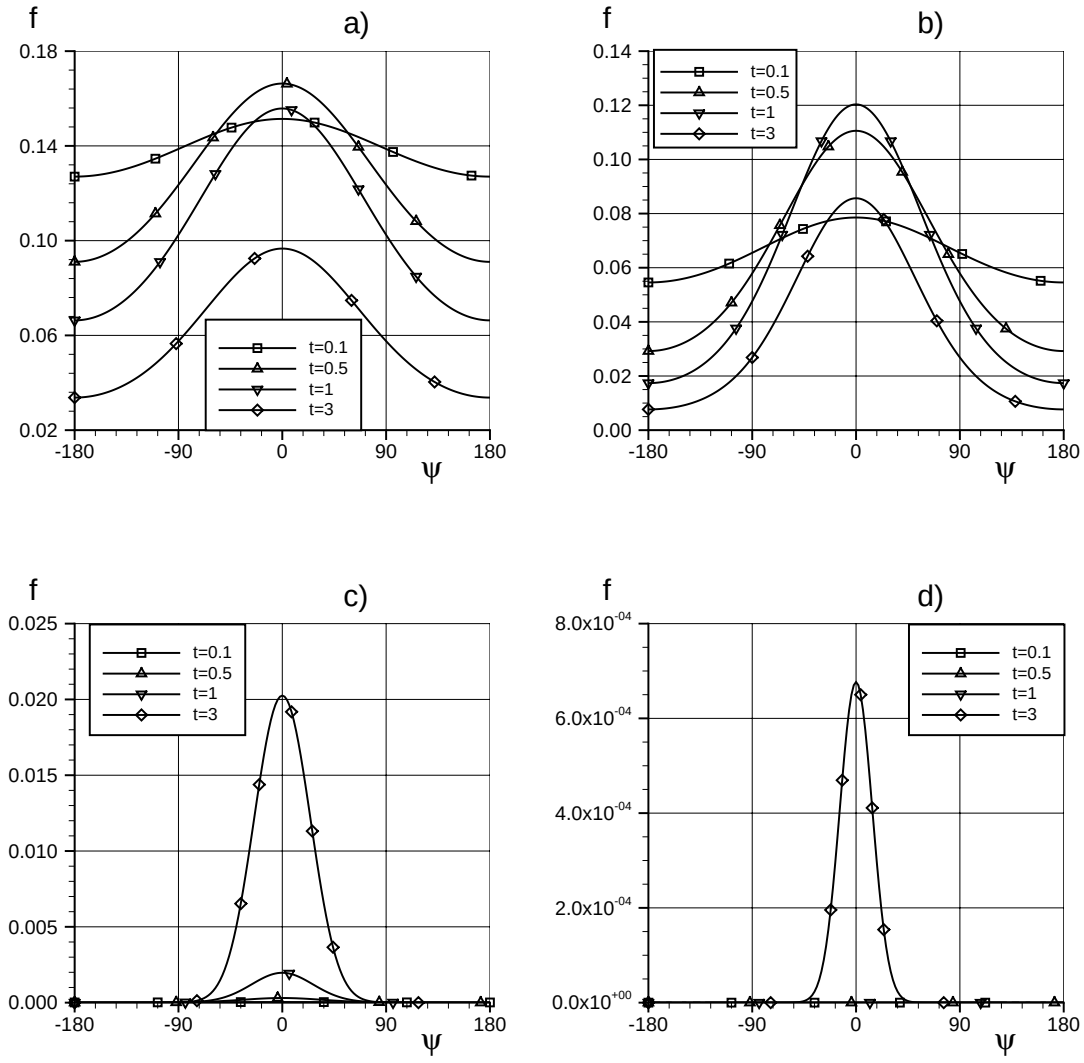

Figure 16: The dependences $f(\psi)$ for different times at different values of the velocity module c for the Coulomb interaction; $\epsilon = 1$, $h = 0$. Figure a) refers to $c = 0.5$; b) – $c = 1$; c) – $c = 3$; d) – $c = 5$.

modified moment method that in which a parameter RT changes in time. A point concerning the optimum choice of a temporal dependence of a parameter RT at different ϵ requires an additional study.

A simpler method was used called "semi-modified moment method". In this method, the calculations of the moment are carried out in an initial basis (temperature of atoms) and, before the DF building-up, a transfer to a new basis is caaried out.

It occurs that, using this method we advance further in studying the running-off

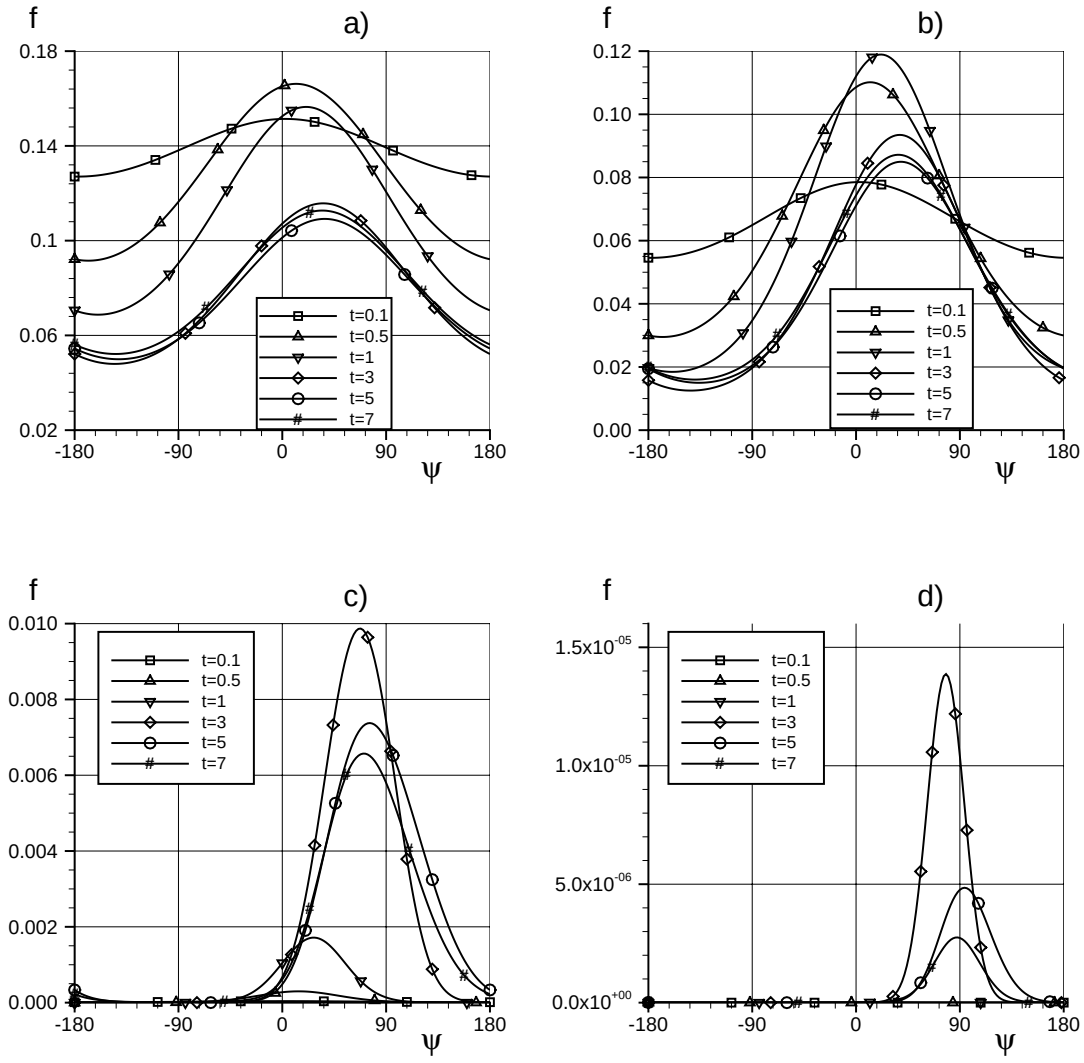

Figure 17: The dependences $f(\psi)$ at different times at different values of the velocity module c for Coulomb interaction; $\epsilon = 1$, $h = 1$. Figure a) refers to $c = 0.5$; b) – $c = 1$; c) – $c = 3$; d) – $c = 5$.

effect in the case of the Coulomb interaction. At $c = 5$, the transfer to a new basis being over, we succeed in description of the running-off process with about one order increase of the maximum on the angular dependence of the DF.

Thus, in Quarter 10, the DF calculations for three interaction models were performed at the different magnetic fields in the domains of the moderate and strong electric field. Our investigations being performed, the results were obtained as follows

1. The DF evolution is strongly dependent on the features of particle interaction.

Especially strongest is an influence of interaction potential on the DF in a domain of high velocities.

2. At zero magnetic field, in a domain of high velocities, in the vicinity $\psi = 0$ there is maximum of which height, at the same electric field, becomes enormously higher when there is a transfer of the hard potential (HS–model) to the softer one (CEM-model).

3. With an increase in the magnetic field, there is a rotation of location of the maximum toward positive ψ, and the maximum height decreases; in a domain of the strong magnetic fields $h \geq 1$, relaxation process of the maximum becomes non-monotonous.

4. At the very strong magnetic field ($h = 5$, $h = 10$), the DF evolution does not depend, in practice, on a mode of the particle interaction but it is determined by the magnetic field.

5. At the DF evolution in the case of the Coulomb interaction and the small values of h, a general process is the ion running-off. In a domain of large c, the DF increases monotonously with no saturation.

6. To study the ion running-off, the semi-modified moment method was developed with the calculation of the moments C_{rl} in an initial basis, and, before calculation of the DF, there is a transfer to a new basis with higher temperature.

7. In the case of the Coulomb interaction and the very strong magnetic field in a domain of high velocities, unusual phenomena were revealed related with the intense oscillations of a cloud of the fast particles.

11 Eleventh quarter

The 11th quarterly report [37] deals with ion diffusion at the electric field.

A small ion admixture being presented at the background of a number of atoms, only transport process exists, i.e., diffusion, besides the transfer of charges under the electric field. To study this process, we use, also, the non- stationary approach. With this aim, at $t = 0$, a concentration gradient ∇n is artificially introduced into initially uniform ion ensemble. This gradient is supposed to be constant in a neighborhood of a point $\mathbf{r_0}$ under consideration, the size of this neighbourhood should be many times the free length path. Solving the Boltzmann equation, the DF evolution is found as well as diffusion coefficient. At sufficiently long time, the process saturation occurs and a stationary value of the diffusion coefficient is found and corresponding DF. The solution of the Boltzmann equation is built up via the moment method with a number of the moments as well as use of all store of knowledge on the ion mobility calculation. At the weak electric field, mobility and diffusion effects do not depend of each other. At the more strong electric field, the diffusion coefficient depends on the electric field value. In this case, two diffusion coefficients are built in: transverse and longitudinal. Independently on the electric field value, as it usually proceeds [4], the concentration gradient value is set as the small one. In this case,

only the first order perturbation theory can be used. Under the sufficiently high electric field, the problem is solved as follows. First, using the developed codes, the problem concerning the ion evolution after a sharp switch-on of the electric field of given value is solved. The stationary DF obtained in this problem is chosen as an initial condition to solve the next problem concerning the DF evolution after the sharp switch-on of ∇n.

In the 11th quarterly report, the known results concerning the ion diffusion and mobility are described in details as well as the various variants of the transfer to the dimensionless variables. It is shown that a dimensionless electric field ϵ_M from [4] is four times ϵ determined by us. Hence, $\epsilon_M = 4$ corresponds to $(\epsilon = 1)$ the latter being the low boundary of the strong field determined by us.

In the dimensionless formulation, the Boltzmann equation for ion diffusion at no electric field takes a form

$$\frac{\partial f}{\partial t} + \delta c_z f = \hat{J}(f, M), \tag{125}$$

and, for ion motion at the electric field (with no diffusion due th concentration gradient) it is as follows

$$\frac{\partial f}{\partial t} + \epsilon \frac{\partial f}{\partial c_z} = \hat{J}(f, M), \tag{126}$$

where $\hat{J}(f, M)$ is the dimensionless collision integral,

$$\delta = \frac{dn/d\varsigma}{n} = \lambda \frac{dn/dz}{n}, \qquad \epsilon = \frac{eE\lambda}{2kT}. \tag{127}$$

Here, a dimensional coordinate $\varsigma = z/\lambda$ is used.

Let, in the problem concerning the mobility, we have the very weak electric field $(\epsilon \ll 1)$, and the very small density gradient $(\delta \ll 1)$ is involved in the problem concerning diffusion. In Report [28], the case of the weak electric field is treated in details. If $(\delta \ll 1)$ and $(\epsilon \ll 1)$, the DF is sought in a form $f_\delta(\mathbf{c}) = M(c)(1 + \delta\varphi_\delta(\mathbf{c}))$, referring to the Chapman-Enskog method, and $f_\epsilon(\mathbf{c}) = M(c)(1 + \epsilon\varphi_\epsilon(\mathbf{c}))$, correspondingly.

If the terms of the second and higher orders are neglected and, in the problem concerning mobility, we write $\partial M/\partial c_z = -2c_z M$ for the derivative with respect to the Maxwellian, from (125) and (126), it follows

$$M \frac{\partial \varphi_\delta}{\partial t} + c_z M = \hat{J}(M\varphi_\delta M), \tag{128}$$

$$M \frac{\partial \varphi_\epsilon}{\partial t} - 2c_z M = \hat{J}(M\varphi_\epsilon, M). \tag{129}$$

Comparing (129) and (128), we conclude that

$$\varphi_\delta(\mathbf{c}, t) = -\frac{1}{2}\varphi_\epsilon(\mathbf{c}, t) \tag{130}$$

It follows that, for the mean dimensionless velocities in the cases under consideration we have

$$\mathbf{u}_\delta = -\frac{1}{2}\mathbf{u}_\epsilon. \tag{131}$$

The dimensionless mobility K and diffusion coefficient D are determined as follows

$$u_\epsilon = K\epsilon, \qquad u_\delta = -D\delta. \tag{132}$$

From (131) and (132), it follows

$$D = \frac{1}{2}K. \tag{133}$$

Then, for the dimensional mobility K_{dim} and diffusion coefficient D_{dim}, we find the Einstein relation

$$D_{dim} = \frac{kT}{e}K_{dim}. \tag{134}$$

Moreover, in a course of a whole non-stationary process, a relationship (130) between φ_δ and φ_ϵ holds, i.e., when obtaining the solution of the mobility problem at small ϵ we obtain, also, the solution of the diffusion problem at small δ. In Report [28], the non-stationary mobility problem was solved via the moment method and $\varphi_\epsilon(\mathbf{c}, t)$ was built up for several interaction models.

Earlier, the stationary solution of the diffusion problem $\varphi_\delta(c, \infty)$ for HS–model, was built up when solving an integral equation in [38]. The calculations were carried out up to $c = 6$. If, in this region, using a formula (130), $\varphi_\delta(c, \infty)$ was expressed via the results of our calculations and involve the corresponding normalization, they coincided totally with the results [38].

Once more, it is confirmation of a high accuracy those data which we obtain when solving the Boltzmann equation via the moment method with the very large number of the MEs.

The study of diffusion at the electric field was performed using the perturbation theory via the two-stage non-stationary method.

The report proves a possibility to use the perturbation theory for the set of the problem being under consideration. It was made on an example when the electric field switch-on developed two stage: first, a sufficiently strong electric field ϵ_0 was switched on, second, along the same direction, a small additional electric field ϵ_1 was switched on being considered as a perturbation. Such a test turns out to be true as there is another mode to obtain the final stationary state when switching on initially, at once, a field $\epsilon_0 + \epsilon_1$.

At the switch-on of the electric field ϵ along z-axis, the moment system, describing the non-stationary process, takes a form

$$\frac{\partial C_{rl}}{\partial t} + \epsilon\left(\frac{2r(l+1)}{2l+3}C_{r-1l+1} - \frac{2l}{2l-1}C_{rl-1}\right) = \sum_{r_1}\Lambda_{r,r1,l}C_{r_1 l}. \tag{135}$$

When the system being solved, we find the dependences $C_{rl}(t)$. As a stationary solution $C_{rl}(\infty)$, it is chosen that at sufficiently large time t. With these moments, the stationary DF is restored at the end of the first stage of the process; it is denoted by $f_0(\mathbf{c})$.

At the second stage, the non-stationary process is under consideration after the switch-on of a concentration δ at $t = 0$. The DF is to be find at this stage as follows

$$f(\mathbf{c}, t) = f_0(\mathbf{c}) + \delta M \varphi_\delta(\mathbf{c}, t), \tag{136}$$

where $\varphi_\delta(\mathbf{c}, 0) = 0$.

In the case of the longitudinal diffusion, a function $\varphi_\delta(\mathbf{c}, t)$ is axially symmetric and, in the moment method, it is expanded in terms of the Legendre polynomials. In the case of the transverse diffusion, the DF losses its axial symmetry and its expansion is carried out in terms of the spherical harmonics with all indices l and m. The moment of a function $M(v)\varphi_\delta(\mathbf{c})$ are denoted by c_{rl} and c_{rlm}, respectively.

After rather cumbersome transformations, the moment equations for second stage were deduced. In the case of the longitudinal diffusion, we have

$$\frac{dc_{rl}}{dt} + \epsilon\Big(\frac{2}{2l+3} r(l+1) c_{r-1l+1} - \frac{2}{2l-1} l c_{rl-1}\Big) = \sum_{r_1} \Lambda_{r,r_1,l} c_{r_1,l} - \Big(\frac{l+1}{2l+3}\Big((l+r+3/2)C^{(0)}_{rl+1} - rC^{(0)}_{r-1l+1}\Big) + \frac{l}{2l-1}\Big(C^{(0)}_{rl-1} - C^{(0)}_{r+1l-1}\Big)\Big), \tag{137}$$

and, in the case of the transverse diffusion,

$$\frac{dc_{rlm}}{dt} + \epsilon\Big(\frac{2}{2l+3} r(l+1) c_{r-1l+1m} - \frac{2}{2l-1} l c_{rl-1m}\Big) = \sum_{r_1} \Lambda_{r,r_1,l} c_{r_1,lm} + \frac{1}{2l+3}\Big((l+r+3/2)C^{(0)}_{rl+1} + rC^{(0)}_{r-1l+1}\Big)\delta_{m,1} - \frac{1}{2l-1}\Big(C^{(0)}_{rl-1} - C^{(0)}_{r+1l-1}\Big)\delta_{m,1} \tag{138}$$

Here, upper index 0 of the moments C_{rl} denotes that those are the moments of a function f_0, i.e., the DF which is developed at the end of the first stage, being the initial state for the second one.

When increasing the electric field, a function $\varphi_\delta(\mathbf{c})$, in the case of the longitudinal diffusion, takes the very large value in a domain of large c_z and, for the solution to be clear, it is more suitable to use a function $\psi_\delta(\mathbf{c})$ which is determined from

$$f(\mathbf{c}, t) = f_0(\mathbf{c})(1 + \delta\psi_\delta(\mathbf{c}, t)). \tag{139}$$

and can be easily expressed via $\varphi_\delta(\mathbf{c})$.

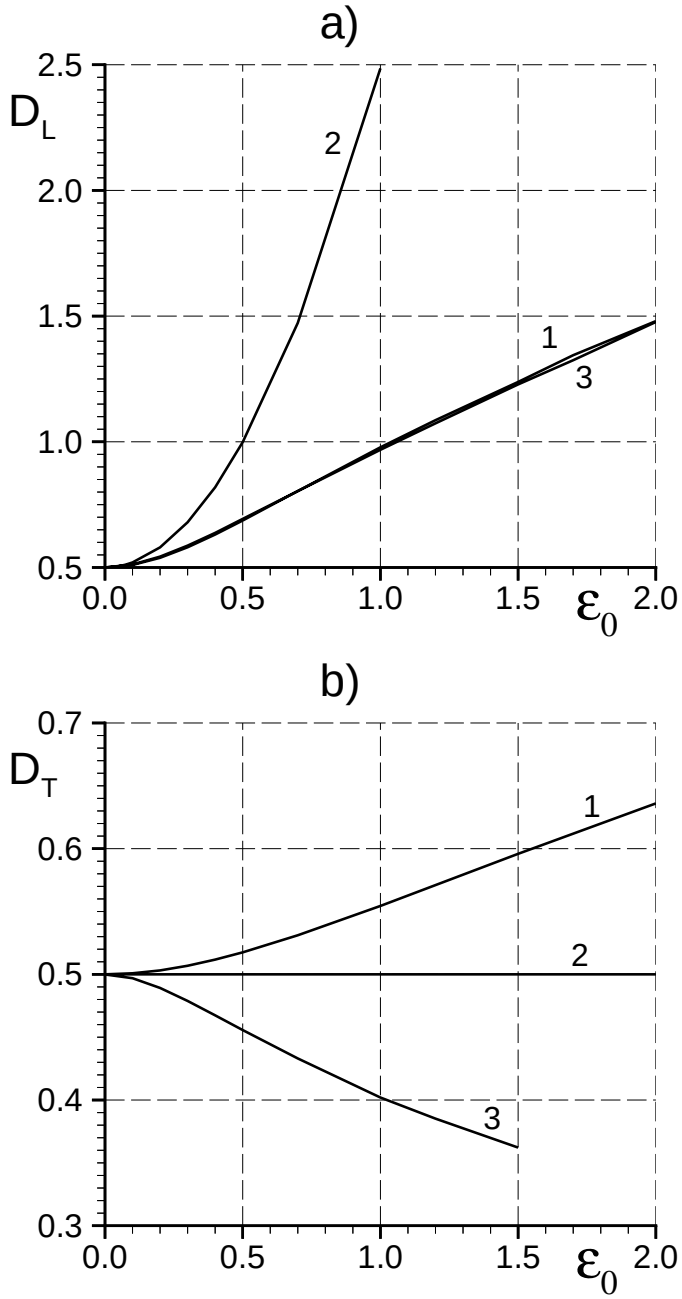

Figure 18: The dependence of the diffusion tensor on the electric field ϵ for the various interaction models. Curve 1 –HS–model, 2 – CEM–model, 3 – CEHS–model. Fig.a – longitudinal diffusion coefficient D_L. Fig.b – transverse diffusion coefficient D_T.

The calculations of the DF evolution were carried out at various values of the electric field ϵ_0 and three interaction models: CEM–model, HS–model, and CEHS–

model. The calculation results were presented in details in the 11th quarterly report [37] in which there are many figures to illustrate the evolution of both $\varphi_\delta(\mathbf{c})$ (or $\psi_\delta(\mathbf{c})$) and the coefficients of the longitudinal and transverse diffusion. Here, we restrict ourselves by unique figure – Fig. 18. In this figure, one can see that the stationary coefficients of the longitudinal D_L and transverse D_T diffusion are strongly influenced by the interaction model.

With an increase in the electric field, for all interaction models, a distinction between D_L and D_T increases. The more pronounced case is CEM–model. Here, D_T does not depend on ϵ at all, and an addition in D_L increases proportionally to ϵ^2.

For HS–model, the transverse diffusion coefficient increases with an increase in ϵ. However, this increase lags sufficiently behind an increase in D_L with an increase in ϵ. In the case of CEHS–model, D_T decreases with an increase in ϵ.

For these two models (HS– and CEHS–models), the dependences $D_L(\epsilon)$ are indistinguishable in practice.

The general results obtained during Quarter 11:

1. To calculate the diffusion at the electric field, the perturbation theory is developed. Along with the stationary state, the non-stationary process developed after the switch-on of a density gradient is calculated.

2. Besides the diffusion coefficient, the perturbation DF is considered. It is shown that, to solve the problem at not so strong electric field, the standard moment method can be successfully used in which the matrix elements of large indices are involved.

3. Any use of the moment method and perturbation theory is confirmed by an example when, instead of the gradient there is a small extra electric field, i.e., under consideration is a two stage process of the electric field switch-on.

4. It is shown that, for an accurate determination of the perturbation DF, the high accuracy knowledge on the stationary DF is required at the electric field. As it is shown in the previous reports, at the strong electric field, any solution of the problem under consideration via the standard moment method meets the well defined difficulties related with the Grad criterion violation.

5. When obtaining the exact stationary solution at the electric field, the fine capabilities of the developed perturbation theory are demonstrated.

6. The calculations of the DF formation after a switch-on of a concentration gradient for several ion-atom interaction models are performed. For the first time, the DF corresponding to a perturbation due to ion diffusion under the electric field is built up.

7. The calculations of the coefficients of the longitudinal and transverse diffusion are carried out at the electric field.

12 Conclusion

This project is based on earlier results concerning the calculations of the matrix elements (ME) of the collision integral of the Boltzmann equation. It was maintained that there is a set of relations between the non-linear moments which are the recurrence relationships. A code package was developed with which aid both the linear and non-linear MEs of the large indices can be found. Earlier, the lack in such MEs does restrain any development of the moment method.

To solve the problem stated in this project, the non-stationary moment method is advanced in which, via solving the non-stationary system of the moment equation, the evolution process is studied after instant switching-off the electric field of taken value. Further, this method can be used to study the ion behaviour at the non-stationary electric field of any form.

The left sides of the moment equations corresponding to the differential part of the Boltzmann equation are introduced. The moment equation are deduced in the global basis. Such approach is called the standard moment method. Also, the equations in the local basis are built up when the parameters of the weighted Maxwellian depend on a coordinate and time. The corresponding method is called the modified moment method.

Considerable attention has been given to the transfer to the dimensionless equations. The standard normalization is determined in which, for any cross- section of ion-atom interaction, the time unit is unambiguously determined. Often, the non-standard normalization is used for which the ion mobility equals unity in the limit of infinitesimally small electric field.

The codes, with which aid the non-stationary moment method is realized, have been developed. These codes are for all problems under consideration in this project. They involve the reading of the ME matrix from the previously calculated Table stored in a special file for taken interaction model. Then, the evolution of the DF moments is calculated and, using them, the DF itself is determined.

The interaction models is advanced for which the analytical solution can be built up. This is CEM–model corresponding to the resonance charge exchange with a cross-section being inversely proportional to the relative velocity. At the electric field switch-on at no magnetic field, the analytical solutions are built up both for the moments and the DF.

Using the analytical solution method, the detailed study of the standard moment method convergence was carried out. It was shown that using the non-stationary approach is capable to advance significantly further in building up the stationary regime as compared with using the iteration method of the solution of the stationary system of the moment equation.

For several interaction models, the detailed calculations of the DF evolution and the physical moments are accomplished after the electric field switch-on.

It was carried out the study of the DF and physical moments in crossed mutually perpendicular electric and magnetic fields, It was found that the transient non-stationary process is accompanied with oscillations of a current along the electric field as well as the Hall current. The calculations were carried out for the Coulomb interaction. At the weak magnetic field, the effect of ion running-off is observed, being rapidly suppressed with an increase in the magnetic field.

The non-stationary moment method is generalized to describe the ion diffusion at the electric field. Two-stage process is considered. The first includes the non-stationary process after the electric filed switch-on. The second is involved after yielding the stationary state at the beginning of which a small concentration gradient is switched on. For several interaction models, the coefficients of the longitudinal and transverse diffusion are calculated.

In all problems under consideration in this project, besides the transport coefficients and physical moments, the DF is built up. For the first time, under investigation is the DF evolution after switching on the electric field.

The studies performed result in a clear determination of the limits of the standard moment method. It is shown that a poor convergence is due to violation of the Grad criterion. The paths are drafted for future development of the method to solve the Boltzmann equation. Certain success can be reached when transferring to the modified moment method. More radical solution of the problems is the expansion in terms of the Sonine polynomials. It is shown that the realization of this new method is possible due to arising the new method of calculation of the MEs of the collision integral.

Both the standard moment method and the mew method cited above can be recommended in solving a whole set of the problems of the gas kinetic theory of gases and plasma. Those include the structure of shock wave as well as the study of the near-wall layers concerning the transport problems and physical-chemical processes.

References

[1] *A.Ya. Ender., I.A. Ender*, *Collision integral of the Boltzmann equation and the moment method.* (SPb State University, SPb, 2003). (in Russian).

[2] A. Ya. Ender, I. A. Ender, Phys. Fluids, V.11, 9 (1000) 2720.

[3] *A. Ya. Ender*, Quarterly Report N^o 1 (1st September – 31st December, 2005) on CRDF Project N RUM1-1500-ST-04. The distribution function and transport properties of charged particles in strong, crossed electric and magnetic fields.

[4] E. A. Mason and E. W. McDaniel, *The mobility and diffusion in Gases* (Wiley, New York, 1973).

[5] E. A. Mason and E. W. McDaniel, *Transport Properties of Ions in Gases* (Wiley, New York, 1988).

[6] R. E. Robson and K. F. Ness, Phys. Rev. A 33 (1986) 2068.

[7] K. F. Ness and R. E. Robson, Phys. Rev. A 34 (1986) 2185.

[8] K. F. Ness, J. Phys. D: Appl. Phys. 27 (1994) 1848.

[9] R. D. White, K. F. Ness and R. E. Robson, Applied Surface Science 192 (2002) 26.

[10] K. Kumar, H. R. Skullerud and R. E. Robson, Aust. J. Phys. 33 (1980) 343.

[11] K. Kumar, Phys. Rep, 112(1084) 319.

[12] K. Kumar, Aust. J. Phys. 20 (1867) 295.

[13] K. Kumar, Aust. J. Phys. 37 (1867) 113.

[14] K. Kumar, Ann. Phys. 37 (1966) 113.

[15] I. Talmi, Helv. Phys. Acta. 25 (1952) 185.

[16] D. Burnett, Proc. London Math. Soc., 40 (1935) 382.

[17] I. A. Ender, A. Ya. Ender Dokl. Akad. Nauk SSSR 193, (1970) 61.

[18] E. Hecke, Math. Ann. 1917, Bd. 78, S. 398.

[19] *A. Ya. Ender*, Quarterly Report N^o 2 (1st January – 28th February, 2005) on CRDF Project N RUM1-1500-ST-04.

[20] Cercignani C. *Theory and application of the Boltzmann equation*, (Edinburgh and London, 1975).

[21] *A. Ya. Ender*, Quarterly Report N^o 3 (1st March – 31st May, 2005) on CRDF Project N RUM1-1500-ST-04. The distribution function and transport properties of charged particles in strong, crossed electric and magnetic fields.

[22] T. Kihara, Rev. of Modern Physics, V.25, N.4 (1953) 844.

[23] I. H. Fertsiger, H. G. Kaper. *Mathematical theory of Transport Processes in Gases*, (North-Holland, Amsterdam, 1972).

[24] P. L. Bhatnagar, E. P. Gross, M. Krook, Phys. Rev., 94 (1954) 511.

[25] *A. Ya. Ender*, Quarterly Report N^o 4 (1st June – 31st August, 2005) on CRDF Project N RUM1-1500-ST-04. The distribution function and transport properties of charged particles in strong, crossed electric and magnetic fields.

[26] *A.Ya. Ender*, Quarterly Report N^o 5 (1st September – 30st November, 2005) on CRDF Project N RUM1-1500-ST-04. The distribution function and transport properties of charged particles in strong, crossed electric and magnetic fields.

[27] *Kolyshkin I. N., Ender A. Ya. and Ender I. A.* Numerical solution of an isotropic relaxation method using the Maxwellian expansion method // Zh. Vychisl. Mat. Mat. Fiz. 1988. Vol. 28. N^o 6. P. 901–916. [USSR Comput. Math. Math. Phys. **28**, 187 (1988)].

[28] *A. Ya. Ender*, Quarterly Report N^o 6 (1st December 2005 – 28th February, 2006) on CRDF Project N RUM1-1500-ST-04. The distribution function and transport properties of charged particles in strong, crossed electric and magnetic fields.

[29] S. Chapman, T. G. Cowling, *The mathematical theory of non-uniform gases*, (Cambridge, 1952).

[30] D. Hilbert, Math. Ann. 72 (1912) 562.

[31] E. Hecke, Math. Zs. 12 (1922) 274.

[32] *A. Ya. Ender*, Quarterly Report N^o 7 (1st March – 31st May, 2006) on CRDF Project N RUM1-1500-ST-04. The distribution function and transport properties of charged particles in strong, crossed electric and magnetic fields.

[33] *R. E. Robson*, Aust. J. Phys. 47 (1994) 279.

[34] *A. Ya. Ender*, Quarterly Report N^o 8 (1st June – 31st August, 2006) on CRDF Project N RUM1-1500-ST-04. The distribution function and transport properties of charged particles in strong, crossed electric and magnetic fields.

[35] *A. Ya. Ender*, Quarterly Report N^o 9 (1st October – 31sh December, 2006) on CRDF Project N RUM1-1500-ST-04. The distribution function and transport properties of charged particles in strong, crossed electric and magnetic fields.

[36] *A. Ya. Ender*, Quarterly Report N^o 10 (1st January – 31st March, 2007) on CRDF Project N RUM1-1500-ST-04. The distribution function and transport properties of charged particles in strong, crossed electric and magnetic fields.

[37] *A. Ya. Ender*, Quarterly Report N^o 11 (1st April – 30th June, 2007) on CRDF Project N RUM1-1500-ST-04. The distribution function and transport properties of charged particles in strong, crossed electric and magnetic fields.

[38] *R.V.Tompson and S.K.Loyalka*, Phys. Fluids **30** (7), 1987, p. 2073.